Chapter 1
Coordinate Systems and Transformation

الانظمه الاحداثيه و التحويلات

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Cartesian (x, y, z) الكارتيزي

cylindrical (r, φ, z) الاسطواني

Spherical (r, φ, θ) كروي

1- Vector Multiplication

الضرب القياسي

1-1- Scalar Product (or dot product)

Consider $A = A_x a_x + A_y a_y + A_z a_z$ and $B = B_x a_x + B_y a_y + B_z a_z$

حساب الضرب القياسي

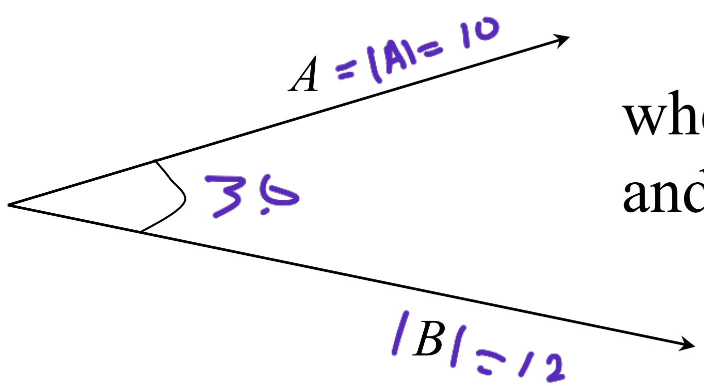
We define the scalar product (or dot product) of vectors A and B as

$$\boxed{\bar{A} \bullet \bar{B} = |A||B|\cos}$$

$|A| \Rightarrow A$ (مقدار ,طول)

$|B| \Rightarrow B$ مقدار ,طول

$$\boxed{|A| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}} \qquad \boxed{|B| = \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2}}$$



where θ is the smaller angle between A and B

$\vec{A} \cdot \vec{B} = 10 \cdot 12 \cdot \cos 45$

1- Vector Multiplication

1-1- Scalar Product (or dot product)

We can prove that $\boxed{\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z}$

Some characteristics of scalar product: $90^\circ =$ متعامدين

♦ Two vectors A and B are said to be orthogonal (or perpendicular) with each other if $\boxed{A \cdot B = 0}$

♦ $\bar{A} \cdot \bar{B} = \bar{B} \cdot \vec{A}$ (commutative law) التبادلية

♦ $\bar{A} \cdot \bar{A} = |A|^2 = A^2$

♦ $\bar{A} \cdot (\bar{B} + \bar{C}) = \bar{A} \cdot \bar{B} + \bar{A} \cdot \bar{C}$ (distributive law) التوزيعية

1- Vector Multiplication

الضرب الاتجاهي

1-2- Vector Product (or Cross Product)

A X B

We define the vector product of vectors A and B as

$$\boxed{A \quad B \qquad |A||B| \sin \quad a_n}$$

where a_n is a unit vector (i.e., $|a_n| \quad 1$) normal to the plane containing A and B

The direction of a_n is taken as the direction of the right thumb when the fingers of the right hand rotate from A to B as shown in figure.

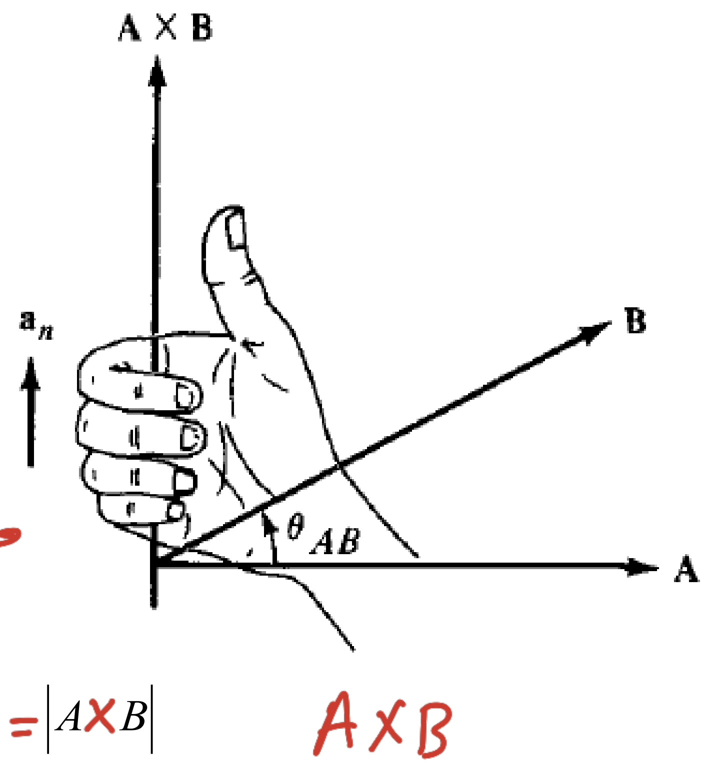
1- Vector Multiplication

1-2- Vector Product (or Cross Product)

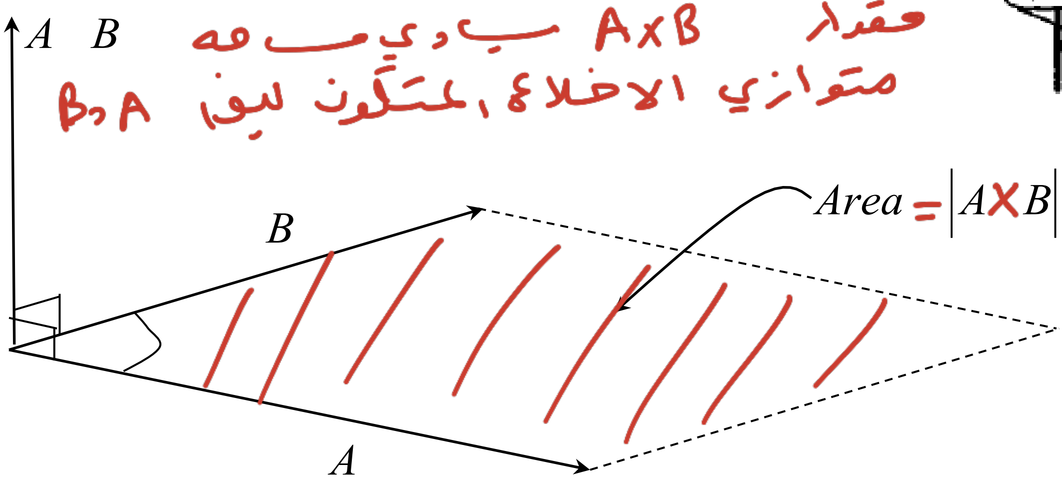
and its magnitude is written as

$$\boxed{|A \times B| = |A||B|\sin\theta}$$

which is the area of the parallelogram formed by *A* and *B* (see Figure)



مقدار A×B بيساوي مساحة
متوازي الاضلاع، المتكون لبين B, A



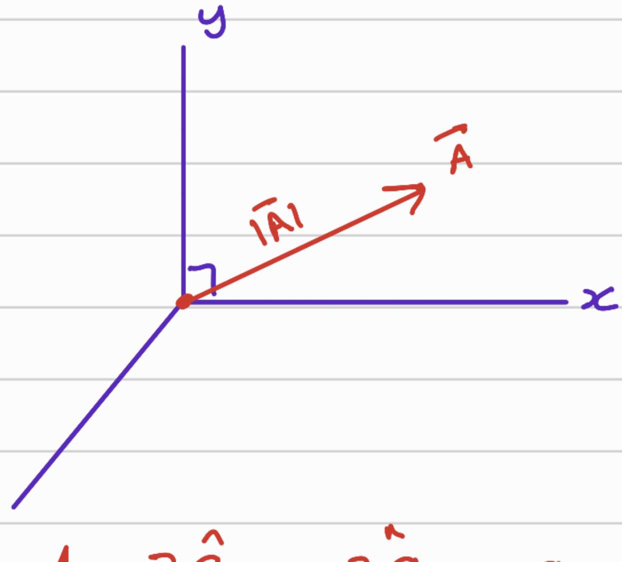
1- Vector Multiplication

1-2- Vector Product (or Cross Product)

We can prove that $A \times B = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

$$A \times B = (A_y B_z - A_z B_y) a_x - (A_x B_z - A_z B_x) a_y + (A_x B_y - A_y B_x) a_z$$

Cartesian Coordinate system (x, y, z)



$A = (A_x, A_y, A_z)$

Component

$A = (3, 2, 1)$

$\vec{A} = 3\hat{x} + 2\hat{y} + \hat{z}$

$A = 3\hat{a}_x + 2\hat{a}_y + a_z$

$\bar{A} = 3\hat{i} + 2\hat{j} + \hat{k}$

$\bar{A} = (A_x, A_y, A_z)$

$\bar{B} = (B_x, B_y, B_z)$

$\bar{A} + \bar{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$

$\bar{A} = (3, 2, 1)$

$\bar{B} = (5, 4, -6)$

$\bar{A} + \bar{B} = (8, 6, -5)$

$8\hat{x} + 6\hat{y} - 5\hat{z}$

والطرح بنفس الطريقة

Magnitude

مقدار أو طول المتجه

$\bar{A} = (3, 2, 1) = 3\hat{x} + 2\hat{y} + 1\hat{z}$

$|\bar{A}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

طرق ضرب المتجهات

1) الضرب بعدد ثابت

الضرب لعدد ثابت $c\vec{A} = (cA_x, cA_y, cA_z)$

$\vec{A} = 3\hat{x} + 2\hat{y} + \hat{z}$ $3\vec{A} = 9\hat{x} + 6\hat{y} + 3\hat{z}$

2) الضرب العياصي scalar product (dot)

$$\vec{A} \bullet \vec{B}$$

نتيجة الضرب هي كمية قياسية (شغل ، زمن)

$\vec{A} = (A_x, A_y, A_z)$ $\vec{B} = (B_x, B_y, B_z)$

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

مثال

$\vec{A} = (3, 1, 5)$ $\vec{B} = (1, -1, 2)$

$$\vec{A} \bullet \vec{B} = 3\times 1 + 1\times(-1) + 5\times 2$$

$$\vec{A} \cdot \vec{B} = 12$$

مثال

$\vec{C} = 2\hat{x} + 3\hat{y} - 4\hat{z}$ $\vec{D} = \hat{x} + 2\hat{y} - k$

$$\vec{C} \cdot \vec{D} = (2\times 1) + (3\times 2) + (-4\times -1) = 12$$

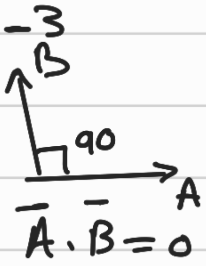
خصائص الضرب النقطي

1 - خاصية تبديلية $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$

2 - الخاصية التوزيعية $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

3- اذا كانت المتجهات متعامدة

$\bar{A} \cdot \bar{B} = 0$



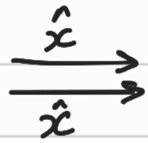
$a_x \cdot a_y = 0$

$a_x \cdot a_z = 0$

$a_y \cdot a_z = 0$

4. يكون الضرب النقطي اكبر ما يمكن عندما تكون الزاوية 0

$\hat{a}_x \cdot \hat{a}_x = 1$



$\hat{a}_y \cdot a_y = 1$

$\hat{a}_z \cdot \hat{a}_z = 1$

.5 $A \cdot A = |A|^2 = A^2$

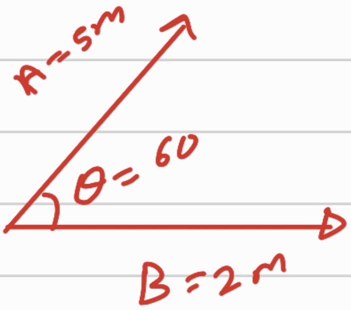
$\vec{A} = 3\hat{a}_x + 2\hat{a}_y + 2\hat{a}_z$

$\bar{A} = 3\hat{a}_x + 2\hat{a}_y + 2\hat{a}_z$

$\bar{A} \cdot \bar{A} = \left(\sqrt{3^2 + 2^2 + 2^2}\right)^2 = 17$

Cross Product

طريقة اخرى كتاب



$$\boxed{\bar{A} \cdot \bar{B} = |A||B| \cos\theta}$$

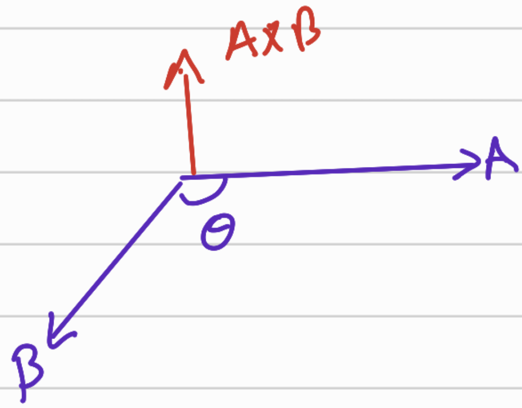
$\bar{A} \cdot \bar{B} = (5)(2)(\cos 60)$

5

الضرب الاتجاهي (التقاطعي)

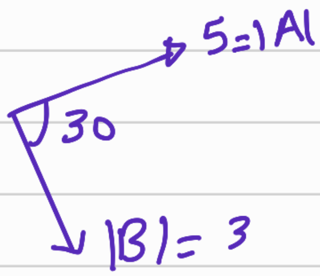
ناتج الضرب يكون متجه له مقدار واتجاه $(A \times B)$

المتجه الناتج من الضرب يكون عمودي على كل من A و B



طريقة ①

$$|A \times B| = |A||B| \sin\theta$$



$$|\vec{A} \times \vec{B}| = 5(3)\sin 30 = 7.5$$

* اذا كان المتجهين متوازيين Parallel
$A \times B = 0$

* اكبر ضرب اتجاهي عندما تكون $\theta = 90$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$(A_yB_z - A_zB_y)\hat{a}_x - (A_xB_z - A_zB_x)\hat{a}_y + (A_xB_y - A_yB_z)\hat{a}_z$$

مثال

$A = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z \qquad B = 4\hat{a}_x + 5\hat{a}_y + 10\hat{a}_z$

$$A \times B = \begin{vmatrix} a_x & a_y & a_z \\ 2 & -3 & 1 \\ 4 & 5 & 10 \end{vmatrix}$$

$$= (-30 - 5)\hat{a}_x - (20 - 4)\hat{a}_y + (10 - -12)\hat{a}_z$$

$$A \times B = -35\hat{a}_x - 16\hat{a}_y + 22\hat{a}_z$$

الضرب الاتجاهي غير تبديلي

$$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$$

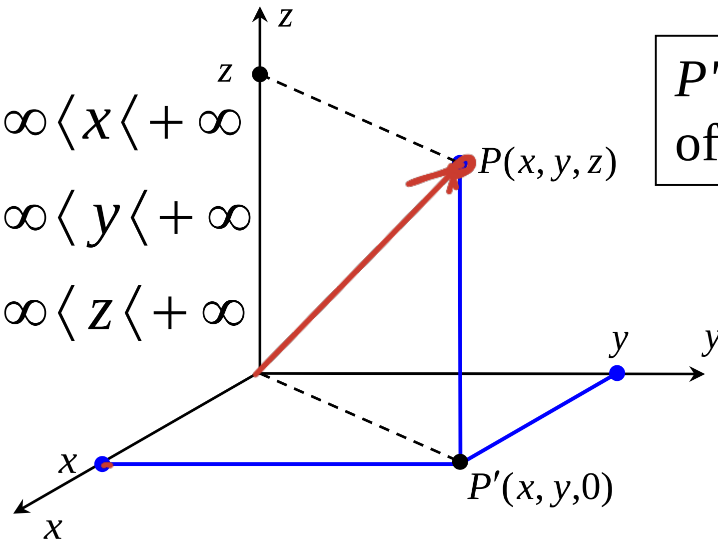
$$a_x \times a_x = \hat{a}_y \times \hat{a}_y = \hat{a}_z \times \hat{a}_z = 0$$

2- Coordinate Systems

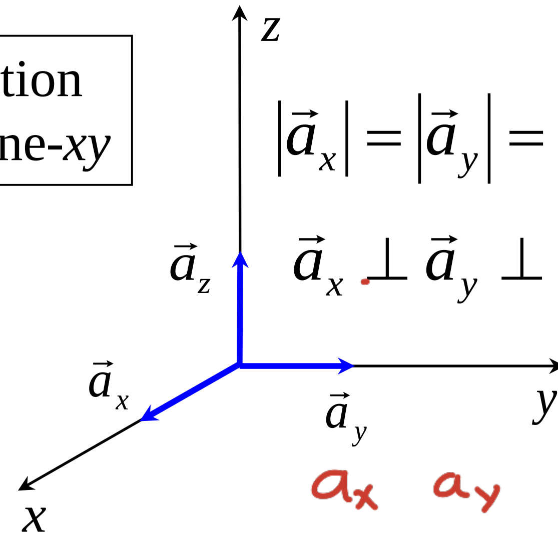
2-1- Cartesian Coordinates (x,y,z)

المستوى، الاحداثي كارتيزي

$$-\infty \langle x \langle +\infty$$
$$-\infty \langle y \langle +\infty$$
$$-\infty \langle z \langle +\infty$$



P' is the projection of P on the plane-xy



$$|\vec{a}_x| = |\vec{a}_y| = |\vec{a}_z| = 1$$

$$\vec{a}_x \perp \vec{a}_y \perp \vec{a}_z$$

$a_x \quad a_y \quad a_z$

متجهات الوحده

طولها = 1

نستخدم للتعبير متجه متعامده

$$\vec{a}_x \cdot \vec{a}_x = \vec{a}_y \cdot \vec{a}_y = \vec{a}_z \cdot \vec{a}_z = 1$$

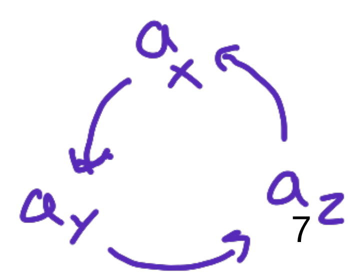
$$\vec{a}_x \cdot \vec{a}_y = \vec{a}_x \cdot \vec{a}_z = \vec{a}_y \cdot \vec{a}_z = 0$$

$$\vec{a}_x \times \vec{a}_x = \vec{a}_y \times \vec{a}_y = \vec{a}_z \times \vec{a}_z = 0$$

$$\vec{a}_x \times \vec{a}_y = \vec{a}_z$$

$$\vec{a}_y \times \vec{a}_z = \vec{a}_x$$

$$\vec{a}_z \times \vec{a}_x = \vec{a}_y$$



A vector $\vec{A}$ in Cartesian coordinates can be written as

$$\vec{A} = (A_x, A_y, A_z) \quad or \quad \vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

2- Coordinate Systems

2-1- Cartesian Coordinates (*x,y,z*)

Magnitude of the vector $\vec{A}$ is written as

مقدار المتجه

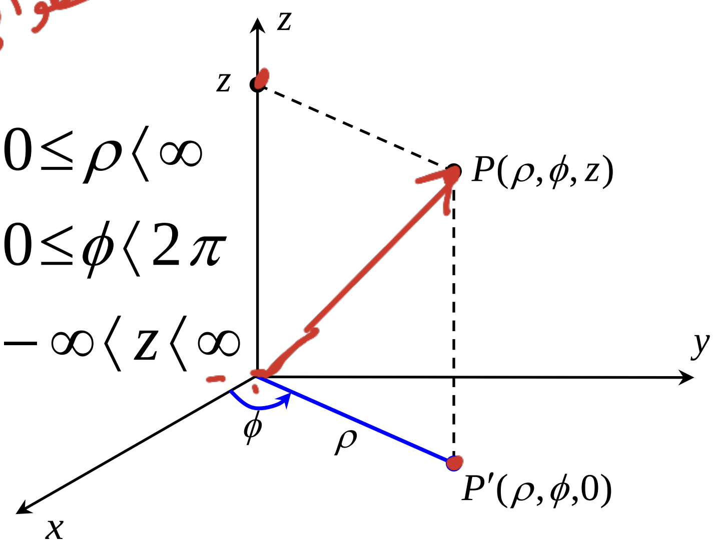
$$\left|\vec{A}\right| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$$

$\vec{A} = 3a_x + 2a_y + 5a_z$

$|\vec{A}| = \sqrt{3^2 + 2^2 + 5^2}$

النظام الإحداثي الاسطواني

2-2- Circular Cylindrical Coordinates (ρ,ϕ,z)



$0 \le \rho \langle \infty$

$0 \le \phi \langle 2\pi$

$-\infty \langle z \langle \infty$

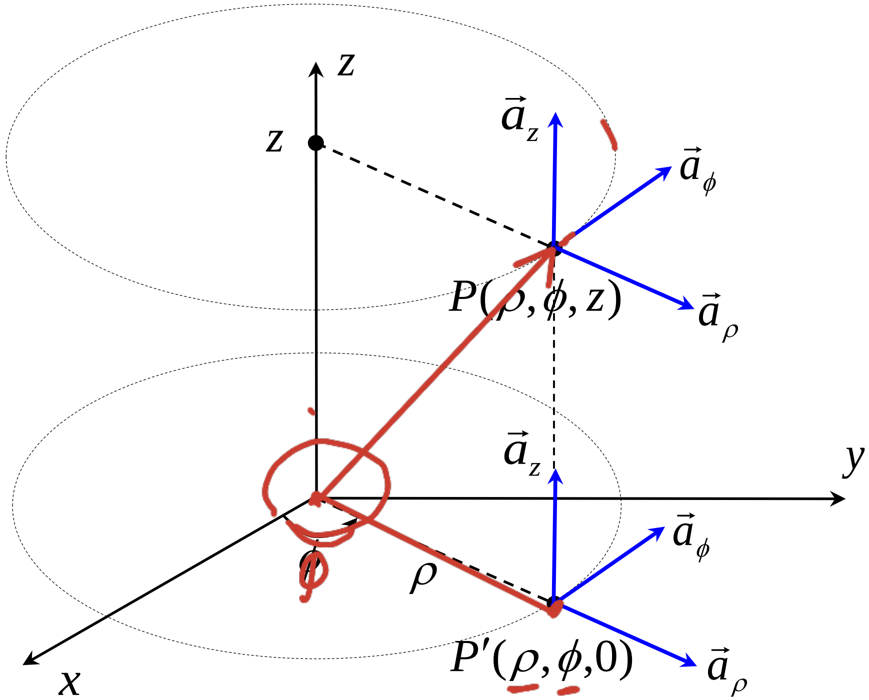
-) ρ is defined as the distance from the origin to point P' or the radius of a cylinder passing through P (the z-axis is its axis of symmetry)

-) ϕ called the azimuthal angle, is measured from the positive *x*-axis taken As reference and the line from origin to P' in the *xy*-plane.

-) *z* is the same as in Cartesian system.

2- Coordinate Systems

2-2- Circular Cylindrical Coordinates (ρ,ϕ,z)



$$\left|\vec{a}_\rho\right| = \left|\vec{a}_\phi\right| = \left|\vec{a}_z\right| = 1$$

$$\vec{a}_\rho \cdot \vec{a}_\rho = \vec{a}_\phi \cdot \vec{a}_\phi = \vec{a}_z \cdot \vec{a}_z = 1$$

$$\vec{a}_\rho \cdot \vec{a}_\phi = \vec{a}_\rho \cdot \vec{a}_z = \vec{a}_\phi \cdot \vec{a}_z = 0$$

$$\vec{a}_\rho \times \vec{a}_\rho = \vec{a}_\phi \times \vec{a}_\phi = \vec{a}_z \times \vec{a}_z = 0$$

$$\vec{a}_\rho \times \vec{a}_\phi = \vec{a}_z$$

$$\vec{a}_\phi \times \vec{a}_z = \vec{a}_\rho$$

$$\vec{a}_z \times \vec{a}_\rho = \vec{a}_\phi$$

ρ ϕ z

$\vec{a}_\rho$ points in the direction of increasing ρ, $\vec{a}_\phi$ in the direction of increasing ϕ, and $\vec{a}_z$ in the positive z-direction.

2- Coordinate Systems

2-2- Circular Cylindrical Coordinates (ρ, ϕ, z)

A vector $\vec{A}$ in cylindrical coordinates can be written as

$$\vec{A} = (A_\rho, A_\phi, A_z) \quad or \quad \vec{A} = A_\rho \vec{a}_\rho + A_\phi \vec{a}_\phi + A_z \vec{a}_z$$

Magnitude of the vector $\vec{A}$ is written as

$$\left|\vec{A}\right| = \sqrt{(A_\rho)^2 + (A_\phi)^2 + (A_z)^2}$$

$A = A_x a_x + A_y a_y + A_z a_z$

$A = A_\rho a_\rho + A_\phi a_\phi + A_z a_z$

2- Coordinate Systems

الكروي

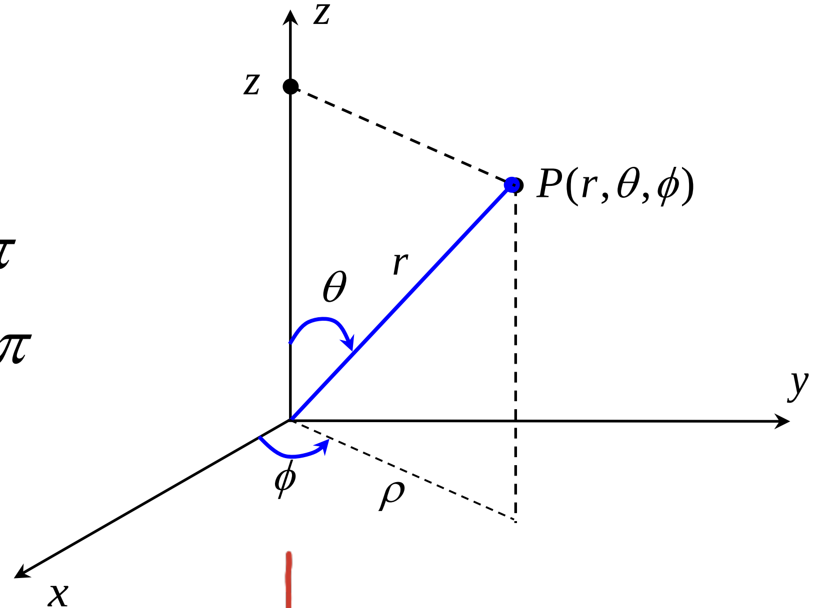
2-3- Spherical Coordinates System (r,θ,ϕ)

٢ ضول متجة

$$0 \le r \langle \infty$$

$$0 \le \theta \le \pi$$

$$0 \le \phi \langle 2\pi$$



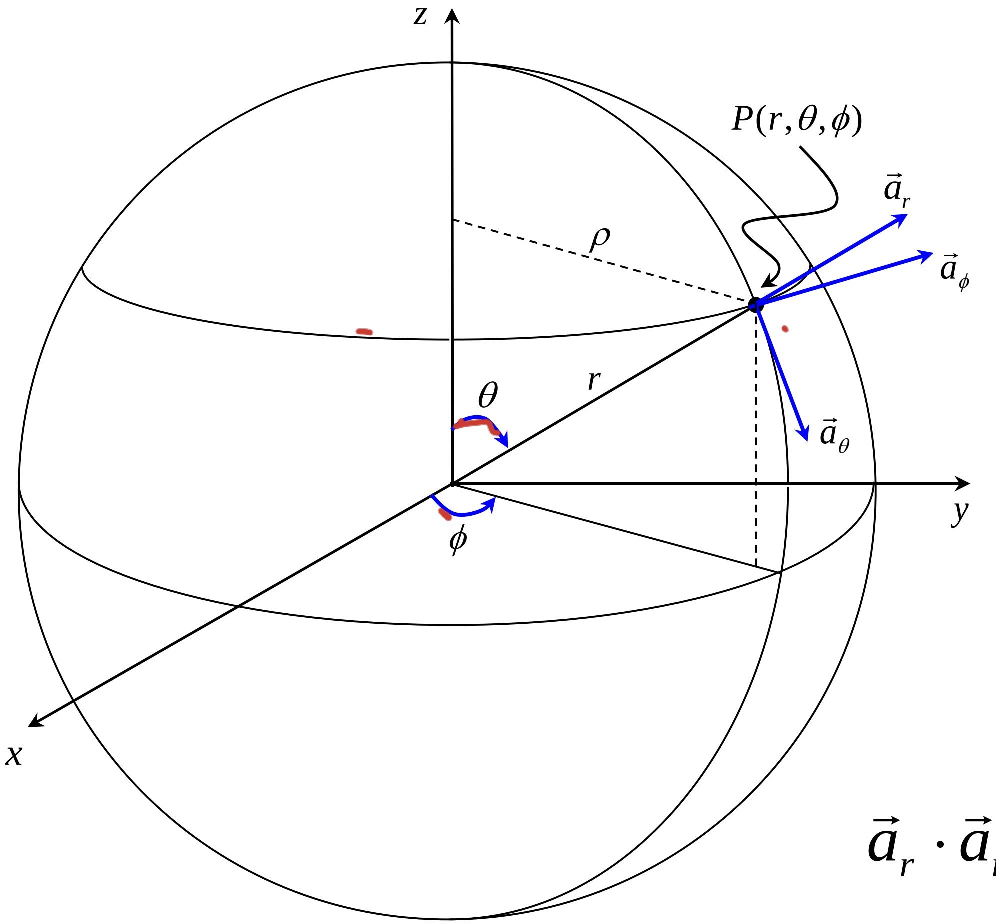
-) ***r*** is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through *P*,

-) $\boldsymbol{\theta}$ (called the colatitude) is the angle between the positive *z*-axis taken as reference and the line from the origin to P,

-) ϕ is the same as defined in cylindrical system.

2- Coordinate Systems

2-3- Spherical Coordinates System (r, θ, ϕ)



$\vec{a}_r$ points in the direction of increasing r,

$\vec{a}_\theta$ points in the direction of increasing θ,

$\vec{a}_\phi$ points in the direction of increasing ϕ.

$$\left|\vec{a}_r\right| = \left|\vec{a}_\theta\right| = \left|\vec{a}_\phi\right| = 1$$

$$\vec{a}_r \perp \vec{a}_\theta \perp \vec{a}_\phi$$

$$\vec{a}_r \cdot \vec{a}_r = \vec{a}_\theta \cdot \vec{a}_\theta = \vec{a}_\phi \cdot \vec{a}_\phi = 1$$

$$\vec{a}_r \cdot \vec{a}_\theta = \vec{a}_r \cdot \vec{a}_\phi = \vec{a}_\theta \cdot \vec{a}_\phi = 0$$

2- Coordinate Systems

2-3- Spherical Coordinates System (r,θ,ϕ)

$$\vec{a}_r \times \vec{a}_r = \vec{a}_\theta \times \vec{a}_\theta = \vec{a}_\phi \times \vec{a}_\phi = 0$$

$$\vec{a}_r \times \vec{a}_\theta = \vec{a}_\phi$$

$$\vec{a}_\theta \times \vec{a}_\phi = \vec{a}_r$$

$$\vec{a}_\phi \times \vec{a}_r = \vec{a}_\theta$$

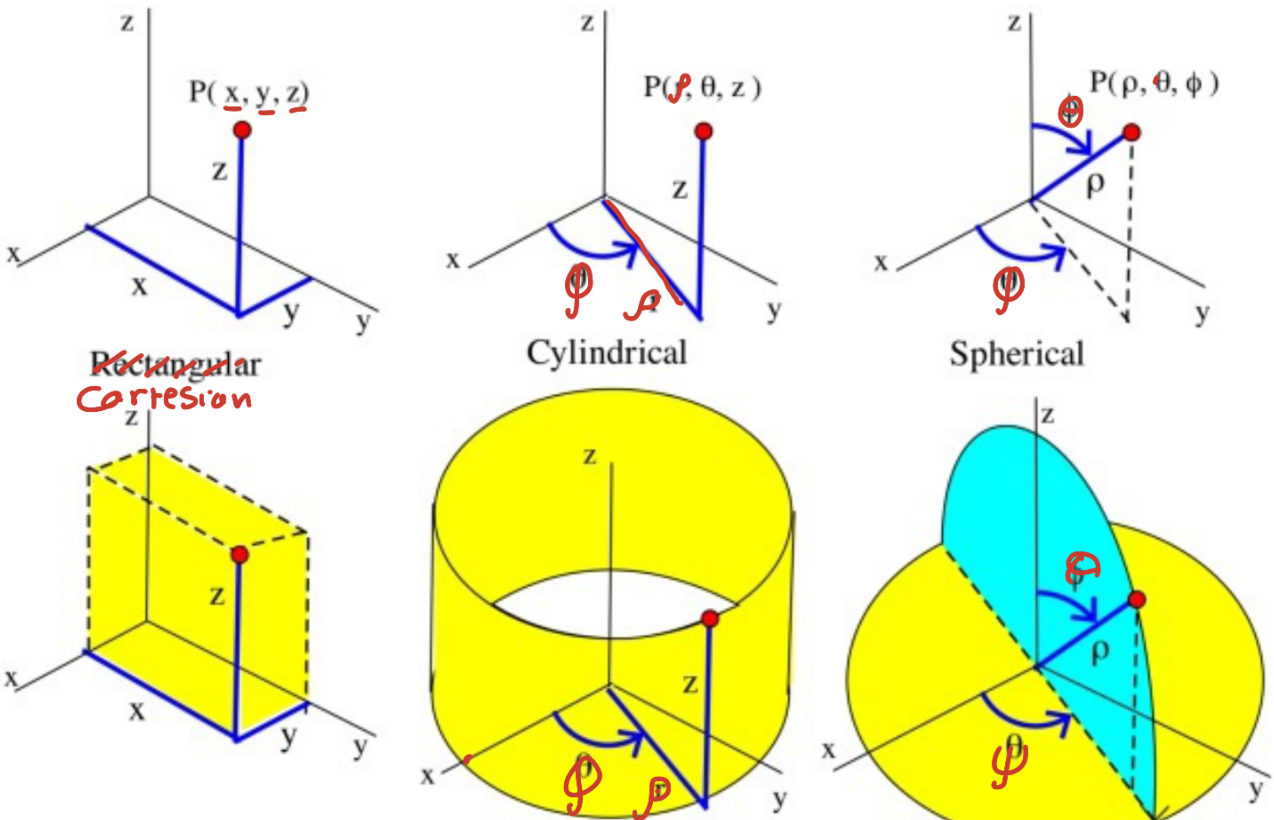
r θ ϕ

A vector $\vec{A}$ in spherical coordinates can be written as

$$\vec{A} = (A_r, A_\theta, A_\phi) \quad or \quad \vec{A} = A_r\vec{a}_r + A_\theta\vec{a}_\theta + A_\phi\vec{a}_\phi$$

Magnitude of the vector $\vec{A}$ is written as

$$\left|\vec{A}\right| = \sqrt{(A_r)^2 + (A_\theta)^2 + (A_\phi)^2}$$



P(x, y, z)
~~Rectangular~~
Cartesion
P(ρ, θ, z)
Cylindrical
P(ρ, θ, ϕ)
Spherical

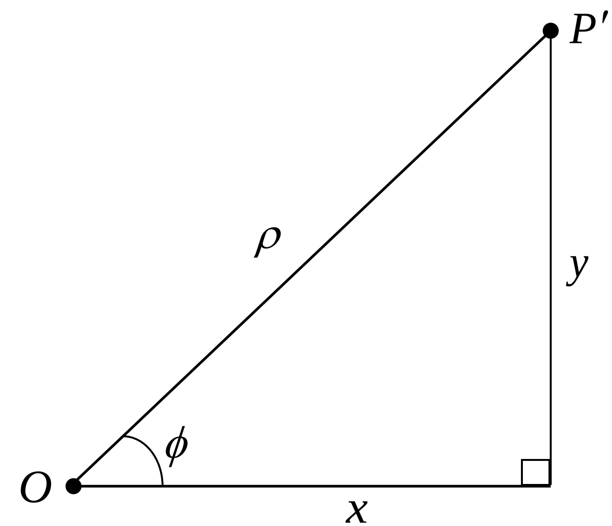
Fig. 2

3- Relationships between Cartesian and cylindrical systems

$$x = \rho\cos\phi, \quad y = \rho\sin\phi$$

$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) + n\pi \quad \text{where n} = 0, 1 \text{ or } 2 \quad \text{and } 0 \le \phi < 2\pi$$

$$\phi = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & \text{if } x > 0 \text{ and } y \ge 0 \\ \tan^{-1}\left(\dfrac{y}{x}\right) + \pi & \text{if } x < 0 \\ \tan^{-1}\left(\dfrac{y}{x}\right) + 2\pi & \text{if } x > 0 \text{ and } y < 0 \\ \dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ \dfrac{3\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$



التحويل بين النظام الكارتيزي و الاسطواني

$$P(x, y, z) \leftarrow P(\rho, \phi, z)$$

لاحاجه لتحويل z

$$x = \rho \cos\phi \qquad y = \rho \sin\phi$$

مثال

Write the cylindrical coordinate $B = (2, \frac{\pi}{6}, 3)$ into Cartesian cor--

$$\begin{array}{c} \rho \quad \phi \quad z \\ B = (2, \frac{\pi}{6}, 3) \end{array} \qquad \begin{array}{c} x \quad\quad y \quad z \\ B = (1.73, 1, 3) \end{array}$$

$$x = \rho \cos\phi = 2 \cos\frac{\pi}{6} = 1.73 \qquad y = \rho \sin\phi = 2 \sin\frac{\pi}{6} = 1$$

$$P(x, y, z) \longrightarrow P(\rho, \phi, z)$$

$$\rho = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

مثال

Write the Cartesian location $(3, 4, 2)$ in cylindrical system

$$(3, 4, 2) \qquad \begin{array}{c} \rho \quad \phi \quad z \\ (5, 53, 2) \end{array}$$

$$\rho = \sqrt{3^2+4^2} = 5$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53$$

ملاحظات حول الزاوية ϕ

اذا كانت x و y موجبة نأخذ الاجابة نفسها

اذا كانت x سالبة نجمع زاوية 180 او π

اذا كانت y سالبة x موجب نجمع 360 او 2π

اذا كانت $x=0$ ← y موجبة $\phi = \pi/2$
← y سالبة $\phi = \frac{3\pi}{2}$

اذا كانت $x=0$ $y=0$ ϕ غير معرفة

(x, y, z) (ρ, ϕ, z)

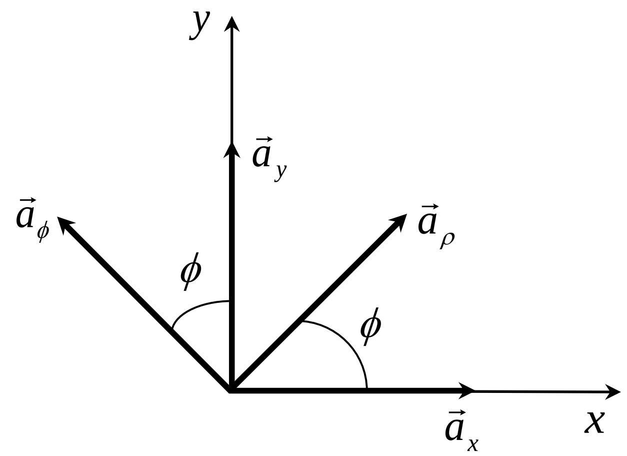
متجهات الوحدة (ρ, ϕ, z) متجهات الوحدة (x, y, z)

$$a_\rho = \cos\phi\, a_x + \sin\phi\, a_y$$
$$a_\phi = -\sin\phi\, a_x + \cos\phi\, a_y$$
$$a_z = a_z$$

$$a_x = \cos\phi\, a_\rho - \sin\phi\, a_\phi$$
$$a_y = \sin\phi\, a_\rho + \cos\phi\, a_\phi$$
$$a_z = a_z$$

3- Relationships between Cartesian and cylindrical systems

$$\vec{a}_x = \cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi$$
$$\vec{a}_y = \sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi$$
$$\vec{a}_z = \vec{a}_z$$



$$\vec{a}_\rho = \cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y$$
$$\vec{a}_\phi = -\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y$$
$$\vec{a}_z = \vec{a}_z$$

<u>Example:</u> Given a point P(-2,6,0) and a vector $\vec{A} = y\vec{a}_x + x\vec{a}_y$

a) Express P and $\vec{A}$ in cylindrical coordinates system

3- Relationships between Cartesian and cylindrical systems

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32m$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) + 180^\circ = \tan^{-1}\left(\frac{6}{-2}\right) + 180^\circ = 108.43^\circ$$

Thus, $P(-2,6,0) = P(6.32, 108.43^\circ, 0)$

$$\vec{A} = y\vec{a}_x + x\vec{a}_y$$

$$\vec{A} = \rho \sin\phi(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi) + \rho\cos\phi(\sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi)$$

$$\vec{A} = \left(\rho\sin\phi\cos\phi + \rho\cos\phi\sin\phi\right)\vec{a}_\rho + (-\rho\sin^2\phi + \rho\cos^2\phi)\vec{a}_\phi$$

3- Relationships between Cartesian and cylindrical systems

b) Evaluate $\vec{A}$ at P in the Cartesian and cylindrical system

In Cartesian system : $\vec{A} = y\vec{a}_x + x\vec{a}_y$

$P(-2,6,0):\quad \vec{A} = 6\vec{a}_x - 2\vec{a}_y$ at P in Cartesian system

In cylindrical system:

$$\vec{A} = \left(\rho \sin\phi \cos\phi + \rho \cos\phi \sin\phi\right)\vec{a}_\rho + (-\rho \sin^2\phi + \rho \cos^2\phi)\vec{a}_\phi$$

At $P(6.32, 108.43^\circ, 0):\quad \rho = 6.32m,\ \sin\phi = \dfrac{6}{\sqrt{40}},\ \cos\phi = \dfrac{-2}{\sqrt{40}}$

$\vec{A} = -3.794\vec{a}_\rho - 5.060\vec{a}_\phi$ at P in cylindrical system

Example: Given a point P(-2,6,0) and a vector $\vec{A} = y\vec{a}_x + x\vec{a}_y$

a) Express P and $\vec{A}$ in cylindrical coordinates system

$$P(\overset{x}{-2}, \overset{y}{6}, \overset{z}{0}) \longrightarrow P(\overset{\rho}{}\ \overset{\phi}{}\ \overset{z}{0})$$

$$\rho = \sqrt{x^2+y^2} = \sqrt{(-2)^2+6^2} = 6.32$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right) = -71.5+180$$
$$= 108.4^\circ$$

$$P(6.32,\ 108.4^\circ,\ 0)$$

$$\bar{A} = y\bar{a}_x + x\bar{a}_y$$

$$\bar{A} = \rho\sin\phi(\cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi) + \rho\cos\phi(\sin\phi\,\hat{a}_\rho + \cos\phi\,\hat{a}_\phi)$$

$$\vec{A} = \rho\sin\phi\cos\phi\,\hat{a}_\rho - \rho\sin^2\phi\,\hat{a}_\phi + \rho\cos\phi\sin\phi\,\hat{a}_\rho + \rho\cos^2\phi\,\hat{a}_\phi$$

$$\vec{A} = (\rho\sin\phi\cos\phi + \rho\cos\phi\sin\phi)\hat{a}_\rho + (-\rho\sin^2\phi + \rho\cos^2\phi)\hat{a}_\phi$$

b) Evaluate $\vec{A}$ at P in the Cartesian and cylindrical system

$$(-2, 6, 0) \qquad\qquad (6.32,\ 108.43,\ 0)$$

cartesian

$$\vec{A} = y\hat{a}_x + x\hat{a}_y = 6\hat{a}_x - 2a_y$$

Cylindrical

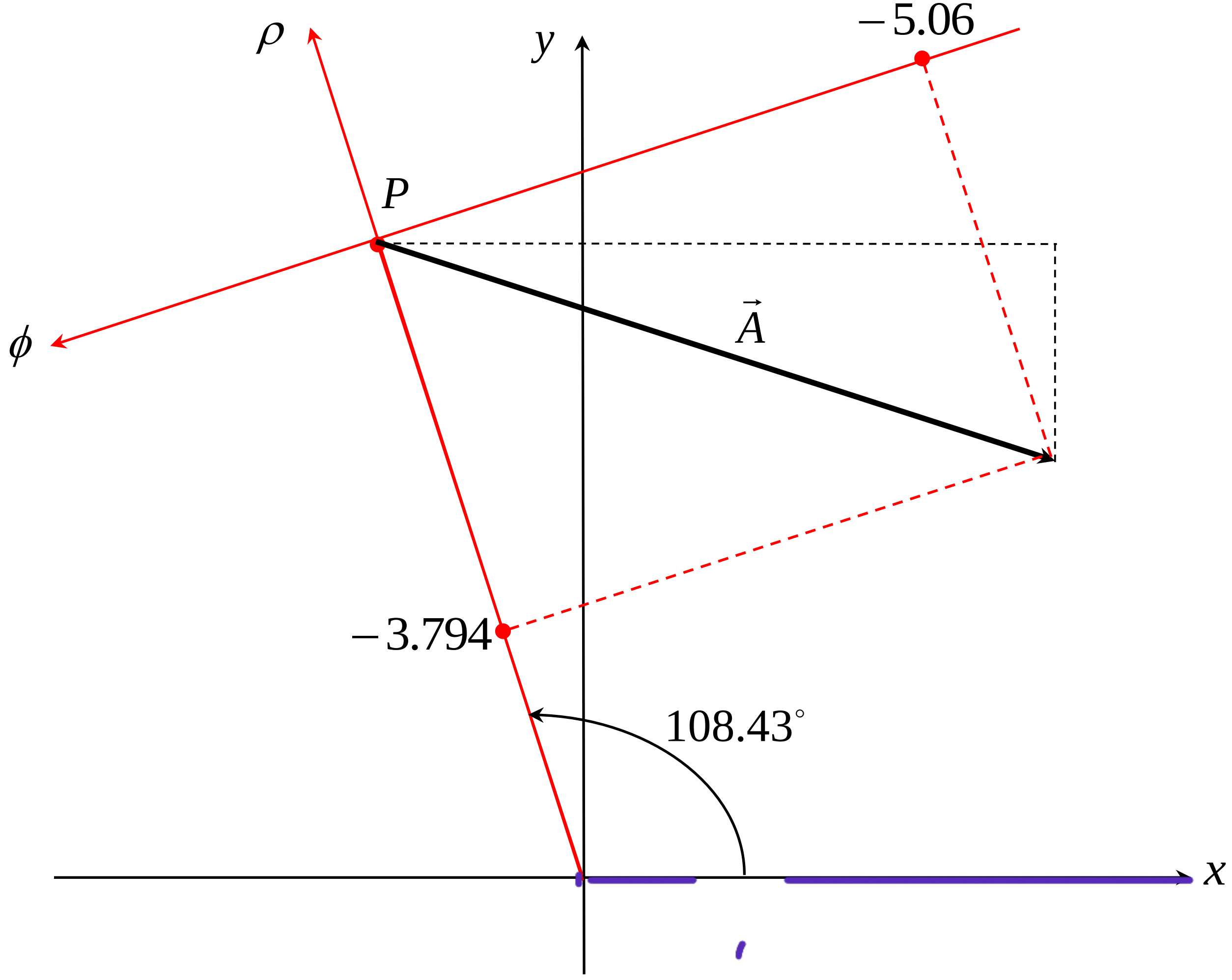
$$\bar{A} = (\underbrace{\rho}_{6.32} \sin \underbrace{\phi}_{108.43} \cos\phi + \rho\cos\phi\sin\phi)\hat{a}_\rho + (-\rho\sin^2\phi + \rho\cos^2\phi)\hat{a}_\phi$$

$2\rho\sin\phi\cos\phi$

$$= 2(6.32\sin 108.43\cos 108.43)\hat{a}_\rho + (-6.32\sin^2 108.43 + 6.32\cos^2 108.43)\hat{a}_\phi$$

$$\bar{A} = -3.744\hat{a}_\rho - 5.06\hat{a}_\phi$$

3- Relationships between Cartesian and cylindrical systems



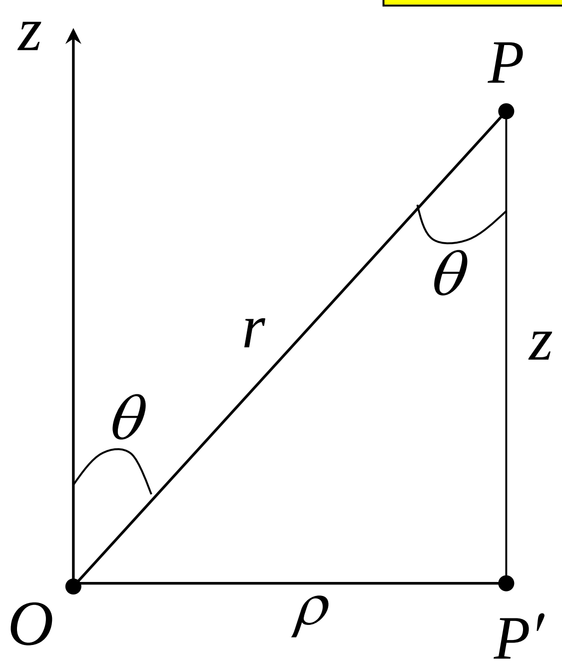
4- Relationships between spherical and cylindrical systems

$$\rho = r\sin\theta$$
$$\phi = \phi$$
$$z = r\cos\theta$$

$$r = \sqrt{\rho^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right) + n\pi, \text{ where } n = 0 \text{ or } 1, \text{ and } 0 \le \theta \le \pi$$

$$\phi = \phi$$

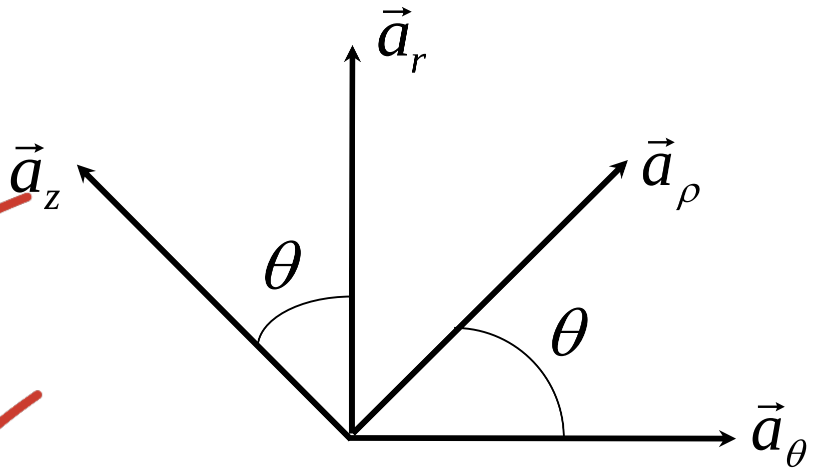


$$\theta = \begin{cases} \tan^{-1}\left(\frac{\rho}{z}\right) & \text{if } z > 0 \\ \tan^{-1}\left(\frac{\rho}{z}\right) + \pi & \text{if } z < 0 \text{ and } \rho \neq 0 \\ \pi & \text{if } z < 0 \text{ and } \rho = 0 \\ \frac{\pi}{2} & \text{if } z = 0 \text{ and } \rho \neq 0 \\ \text{Undefined} & \text{if } z = 0 \text{ and } \rho = 0 \end{cases}$$

4- Relationships between spherical and cylindrical systems

$$\vec{a}_\rho = \sin\theta\,\vec{a}_r + \cos\theta\,\vec{a}_\theta$$
$$\vec{a}_\phi = \vec{a}_\phi$$
$$\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta$$

$$\vec{a}_r = \sin\theta\,\vec{a}_\rho + \cos\theta\,\vec{a}_z$$
$$\vec{a}_\theta = \cos\theta\,\vec{a}_\rho - \sin\theta\,\vec{a}_z$$
$$\vec{a}_\phi = \vec{a}_\phi$$



التحويل من جروية اسطوانية و العكس

$$(r, \phi, \theta) \longrightarrow (\rho, \phi, z)$$

$$\rho = r\sin\theta \qquad a_\rho = \sin\theta\,\hat{a}_r + \cos\theta\,\hat{a}_\theta$$
$$z = r\cos\theta \qquad a_z = \cos\theta\,\hat{a}_r - \sin\theta\,\hat{a}_\theta$$
$$\phi = \phi \qquad a_\phi = a_\phi$$

$$(\rho, \phi, z) \longrightarrow (r, \phi, \theta)$$

$$r = \sqrt{\rho^2 + z^2} \qquad a_r = \sin\theta\,\hat{a}_\rho + \cos\theta\,\hat{a}_z$$
$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right) \qquad a_\theta = \cos\theta\,\hat{a}_\rho - \sin\theta\,\hat{a}_z$$
$$\phi = \phi \qquad \hat{a}_\phi = \hat{a}_\phi$$

Example :-

Convert from cylindrical to spherical

$$\overset{\rho}{(1},\ \overset{\phi}{\frac{\pi}{2}},\ \overset{z}{1)} \longrightarrow \overset{r}{(\sqrt{2}},\ \overset{\phi}{\frac{\pi}{2}},\ \overset{\theta}{45^\circ)}$$

$$r = \sqrt{\rho^2 + z^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} = 45$$

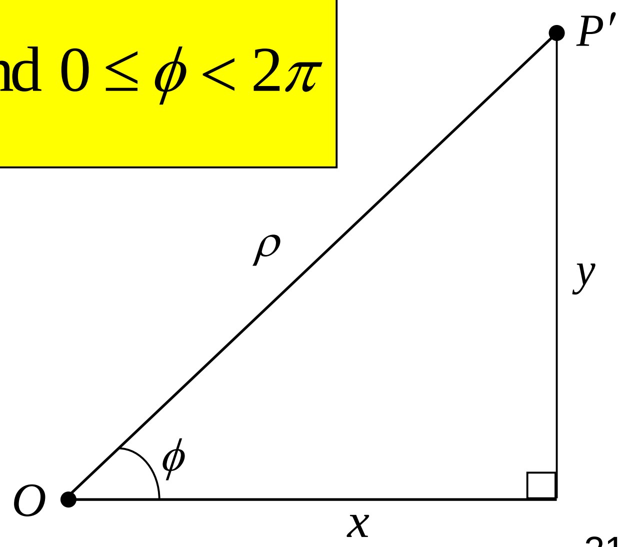
5- Relationships between spherical and Cartesian systems

$$\rho^2 = x^2 + y^2 \qquad r = \sqrt{\rho^2 + z^2}$$

$$\boxed{r = \sqrt{x^2 + y^2 + z^2}}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) + n\pi \text{ where } n = 0 \text{ or } 1\text{; and } 0 \le \theta \le \pi}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{y}{x}\right) + n\pi \text{ where } n = 0 \text{ or } 1 \text{ or } 2\text{; and } 0 \le \phi < 2\pi}$$



التحويل من نظام كارتيزي الى كروي والعكس

$$(x, y, z) \longrightarrow (r, \varphi, \theta)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\hat{a}_r = \sin\theta\cos\varphi\,\hat{a}_x + \sin\theta\sin\varphi\,\hat{a}_y + \cos\theta\,\hat{a}_z$$

$$\hat{a}_\theta = \cos\theta\cos\varphi\,\hat{a}_x + \cos\theta\sin\varphi\,\hat{a}_y - \sin\theta\,\hat{a}_z$$

$$\hat{a}_\varphi = -\sin\varphi\, a_x + \cos\varphi\, a_y$$

التحويل من كروي الى كارتيزي

$$(\rho, \varphi, \theta) \longrightarrow (x, y, z)$$

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

$$\hat{a}_x = \sin\theta\cos\varphi\,\hat{a}_r + \cos\theta\cos\varphi\,\hat{a}_\theta - \sin\varphi\,\hat{a}_\varphi$$

$$\hat{a}_y = \sin\theta\sin\varphi\,\hat{a}_r + \cos\theta\sin\varphi\,\hat{a}_\theta + \cos\varphi\,\hat{a}_\varphi$$

$$a_z = \cos\theta\,\hat{a}_r - \sin\theta\, a_\theta$$

5- Relationships between spherical and Cartesian systems

$$\theta = \begin{cases} \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) & if\ \ z>0 \\ \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)+\pi & if\ \ z<0\ and\ \sqrt{x^2+y^2}\neq 0 \\ \pi & if\ \ z<0\ and\ \sqrt{x^2+y^2}=0 \\ \dfrac{\pi}{2} & if\ \ z=0\ and\ \sqrt{x^2+y^2}\neq 0 \\ Undefined & if\ \ z=0\ and\ \sqrt{x^2+y^2}=0 \end{cases}$$

$$\phi = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & if\ \ x>0\ and\ \ y\geq 0 \\ \tan^{-1}\left(\dfrac{y}{x}\right)+\pi & if\ \ x<0 \\ \tan^{-1}\left(\dfrac{y}{x}\right)+2\pi & if\ \ x>0\ and\ \ y<0 \\ \dfrac{\pi}{2} & if\ \ x=0\ and\ \ y>0 \\ \dfrac{3\pi}{2} & if\ \ x=0\ and\ \ y<0 \\ undefined & if\ \ x=0\ and\ \ y=0 \end{cases}$$

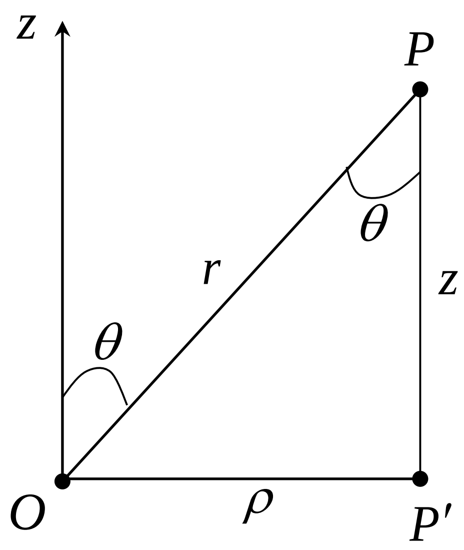
5-Relationships between spherical and Cartesian systems

$$x = \rho\cos\phi \qquad \rho = r\sin\theta$$

$$\boxed{x = r\sin\theta\cos\phi}$$

$$y = \rho\sin\phi \qquad \rho = r\sin\theta$$

$$\boxed{y = r\sin\theta\sin\phi} \qquad \boxed{z = r\cos\theta}$$



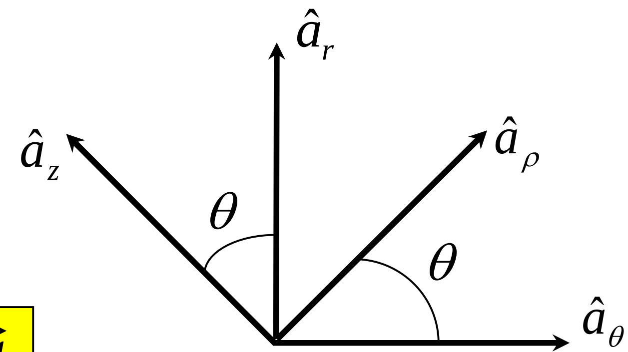
5-Relationships between spherical and Cartesian systems

$$\vec{a}_r = \sin\theta\,\vec{a}_\rho + \cos\theta\,\vec{a}_z$$

$$\vec{a}_\rho = \cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y$$

$$\vec{a}_r = \sin\theta\,(\cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y) + \cos\theta\,\vec{a}_z$$

$$\boxed{\vec{a}_r = \sin\theta\cos\phi\,\vec{a}_x + \sin\theta\sin\phi\,\vec{a}_y + \cos\theta\,\vec{a}_z}$$

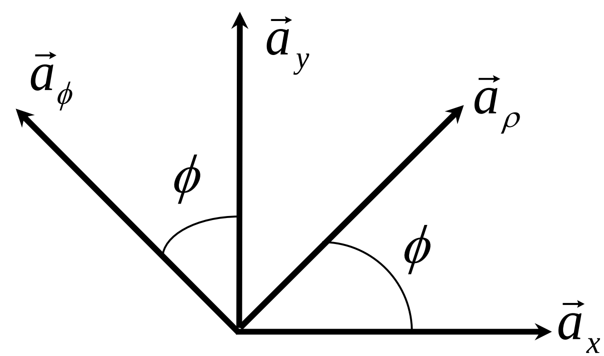


$$\vec{a}_\theta = \cos\theta\,\vec{a}_\rho - \sin\theta\,\vec{a}_z$$

$$\vec{a}_\rho = \cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y$$

$$\vec{a}_\theta = \cos\theta(\cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y) - \sin\theta\,\vec{a}_z$$

$$\boxed{\vec{a}_\theta = \cos\theta\cos\phi\,\vec{a}_x + \cos\theta\sin\phi\,\vec{a}_y - \sin\theta\,\vec{a}_z}$$

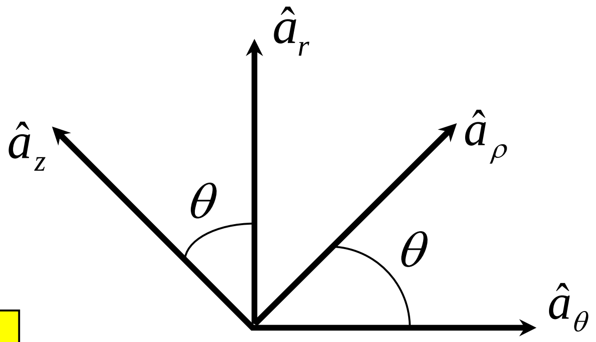


$$\boxed{\vec{a}_\phi = -\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y}$$

5-Relationships between spherical and Cartesian systems

$$\vec{a}_x = \cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi$$

$$\vec{a}_\rho = \sin\theta\,\vec{a}_r + \cos\theta\,\vec{a}_\theta$$

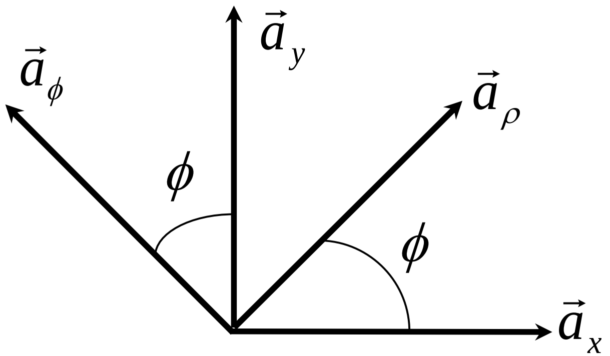


$$\vec{a}_x = \cos\phi\,(\sin\theta\,\vec{a}_r + \cos\theta\,\vec{a}_\theta) - \sin\phi\,\vec{a}_\phi$$

$$\boxed{\vec{a}_x = \sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi}$$

$$\vec{a}_y = \sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi$$

$$\vec{a}_\rho = \sin\theta\,\vec{a}_r + \cos\theta\,\vec{a}_\theta$$



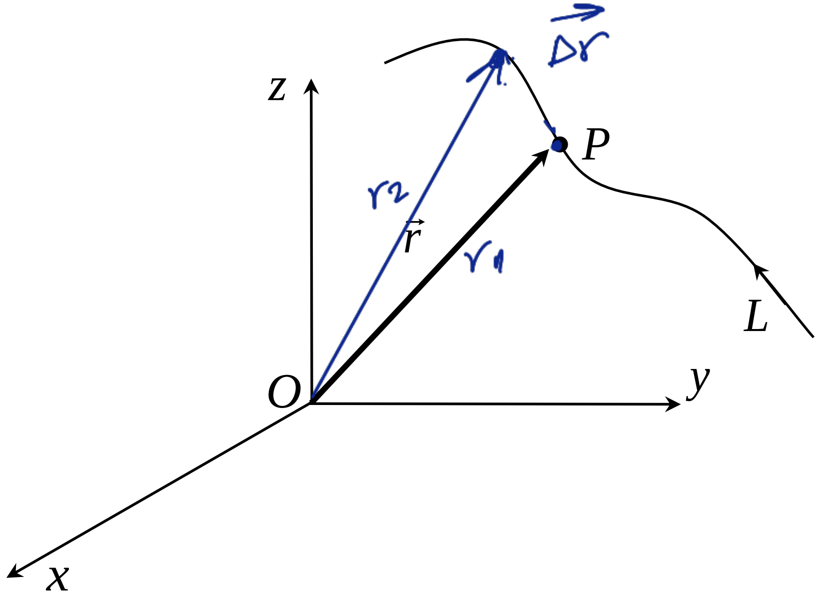
$$\vec{a}_y = \sin\phi\,(\sin\theta\,\vec{a}_r + \cos\theta\,\vec{a}_\theta) + \cos\phi\,\vec{a}_\phi$$

$$\boxed{\vec{a}_y = \sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi}$$

$$\boxed{\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta}$$

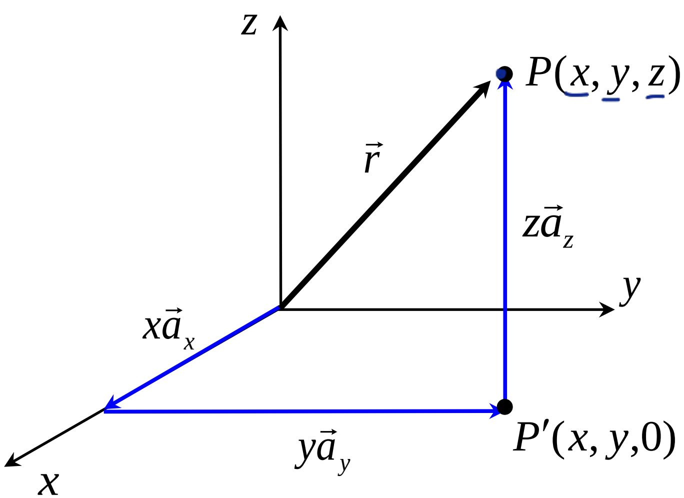
6-Vector position and Differential element in length

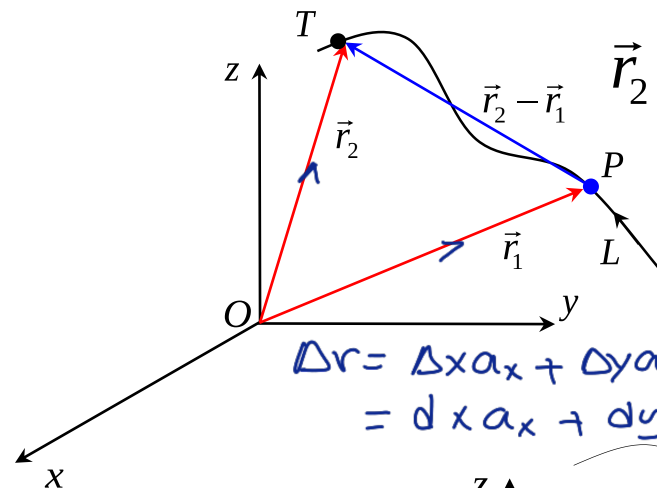
Consider a point P located in space, and consider the direct arrow extending from the origin to this point P. This arrow is known as the position vector $\vec{r}$. A position vector is an alternative way to denote the location of a point P in space.



6- Vector position and Differential element in length

<u>3-1- Cartesian coordinates system</u>





$$\vec{r}_2 - \vec{r}_1 = \Delta\vec{r}$$

$\Delta r = \Delta x a_x + \Delta y a_y + \Delta z \hat{a}_z$
$= dx\, a_x + dy\, a_y + dz\, \hat{z}$

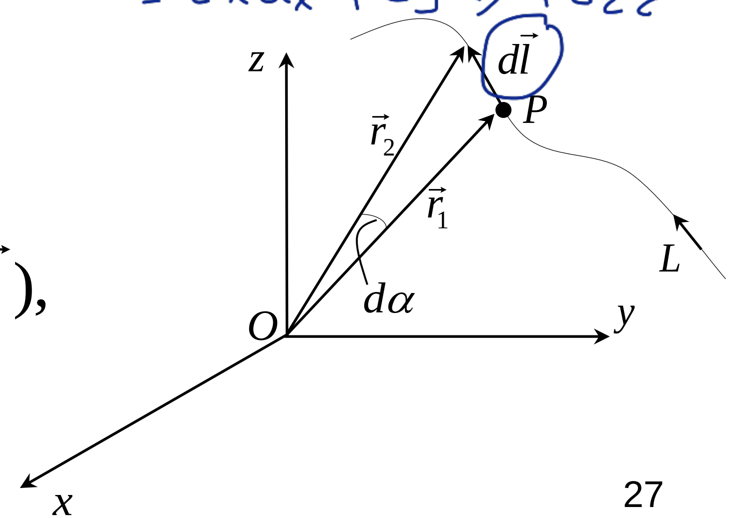
$$\boxed{\vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z}$$

$r_2 = x_2 a_x + y_2 a_y + z_2 \hat{a}$

When $\vec{r}_2 - \vec{r}_1 = \Delta\vec{r} \to 0$ (*i.e.*, $\Delta\vec{r} = d\vec{r} = d\vec{l}$),

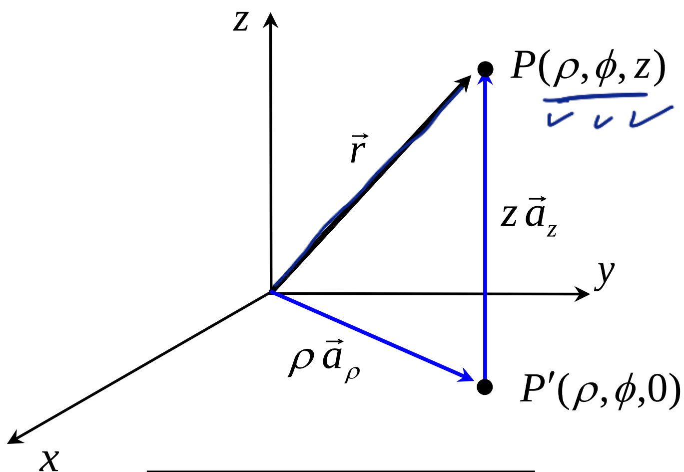
$$\boxed{d\vec{r} = d\vec{l} = dx\vec{a}_x + dy\vec{a}_y + dz\vec{a}_z}$$

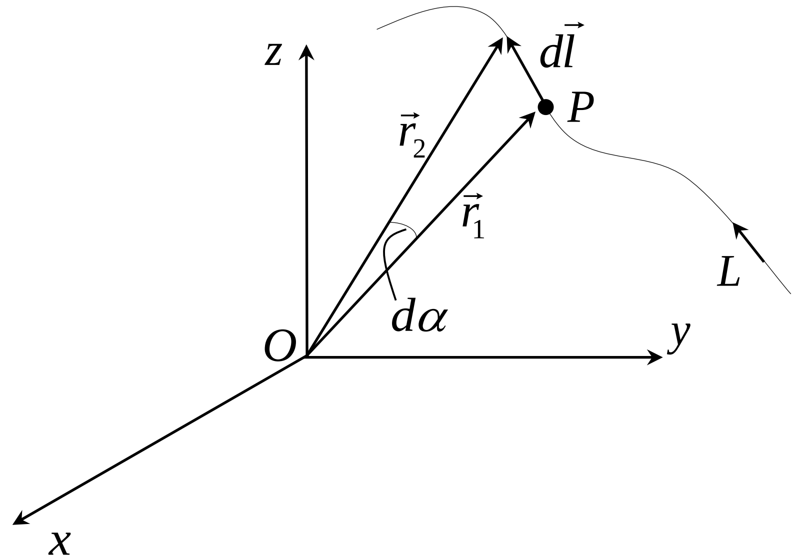
المشتقه في حاله اللامتناهيات الصغر، تجزيه



6-Vector position and Differential element in length

3-2- Circular cylindrical coordinates system





$$\boxed{\vec{r} = \rho\,\vec{a}_\rho + z\,\vec{a}_z}$$

$$d\vec{r} = d\vec{l} = d\rho\,\vec{a}_\rho + \rho\,d\vec{a}_\rho + dz\,\vec{a}_z \qquad \vec{a}_\rho = \cos\phi\,\vec{a}_x + \sin\phi\,\vec{a}_y$$

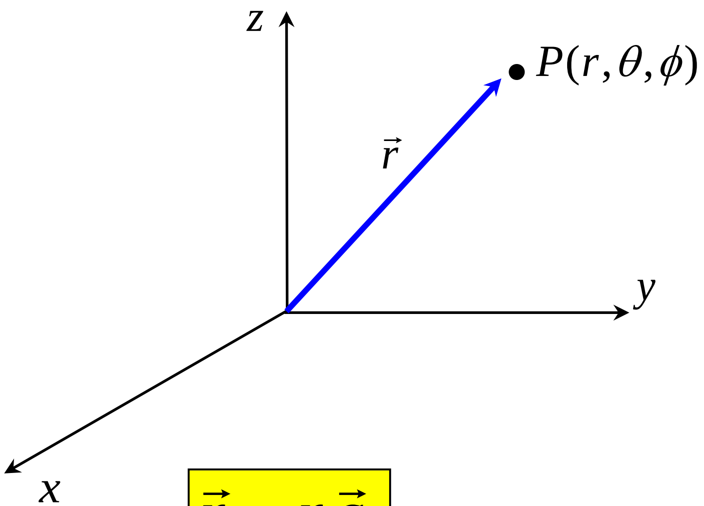
$$\frac{d\vec{a}_\rho}{d\phi} = -\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y = \vec{a}_\phi \quad \Rightarrow \quad d\vec{a}_\rho = d\phi\,\vec{a}_\phi$$

$$\boxed{d\vec{r} = d\vec{l} = d\rho\,\vec{a}_\rho + \rho\,d\phi\,\vec{a}_\phi + dz\,\vec{a}_z}$$

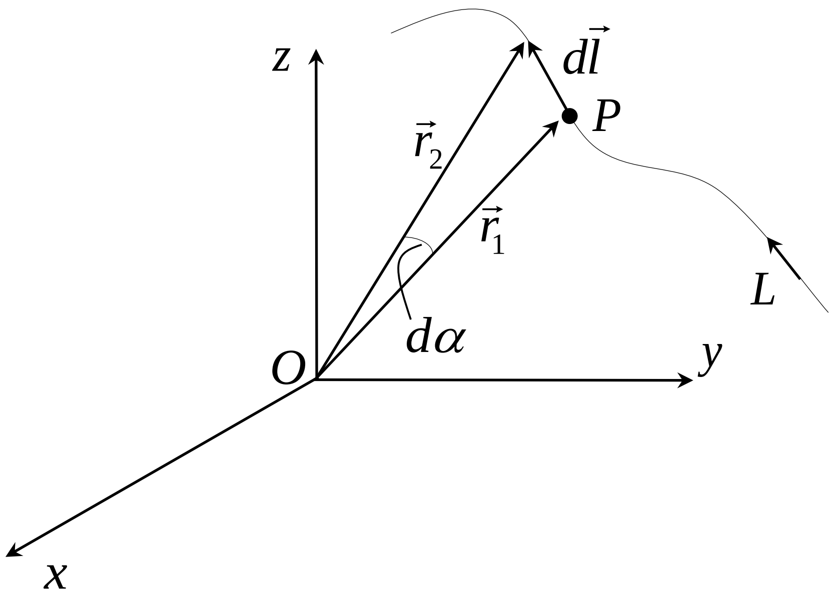
المشتقة في حالة الاحداثيات الاسطوانية

6-Vector position and Differential element in length

3-3- Spherical coordinates system



$$\boxed{\vec{r} = r\,\vec{a}_r}$$



$$\vec{a}_r(\theta,\phi) = \sin\theta\cos\phi\,\vec{a}_x + \sin\theta\sin\phi\,\vec{a}_y + \cos\theta\,\vec{a}_z$$

$$d\vec{a}_r = \frac{\partial\vec{a}_r}{\partial\theta}d\theta + \frac{\partial\vec{a}_r}{\partial\phi}d\phi$$

6-Vector position and Differential element in length

$$d\vec{a}_r = (\cos\theta\cos\phi\,\vec{a}_x + \cos\theta\sin\phi\,\vec{a}_y - \sin\theta\,\vec{a}_z)d\theta +$$
$$(-\sin\theta\sin\phi\,\vec{a}_x + \sin\theta\cos\phi\,\vec{a}_y)d\phi$$

$$d\vec{a}_r = d\theta(\cos\theta\cos\phi\,\vec{a}_x + \cos\theta\sin\phi\,\vec{a}_y - \sin\theta\,\vec{a}_z) +$$
$$\sin\theta\, d\phi(-\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y)$$

$$d\vec{a}_r = d\theta\,\vec{a}_\theta + \sin\theta\, d\phi\,\vec{a}_\phi$$

$$\boxed{d\vec{r} = d\vec{l} = dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi}$$ ✓

المشتقة في حالة الاحداثيات الكروية

Defferential element in length

	متجه الموقع	المشتقة الاولى
Cartesian	$\bar{r} = x\hat{a}_x + y\bar{a}_y + \bar{z}a_z$	$dl = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$
Cylindrical	$\bar{r} = \rho\hat{a}_\rho + z\hat{a}_z$	$dl = d\rho\, a_\rho + \rho\, d\phi\, \hat{a}_\phi + dz\,\hat{a}_z$
spherical	$\bar{r} = r\,\bar{a}_r$	$dl = dr\,a_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\theta\,\hat{a}_\phi$

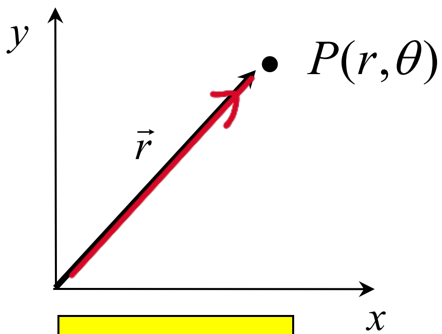
تعديل

$d\phi$

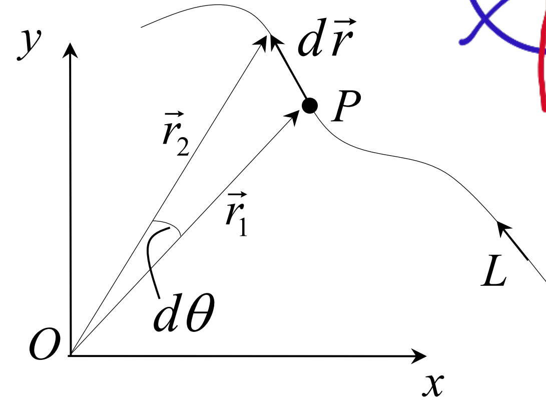
7-Vector Position and Differential Element in Length

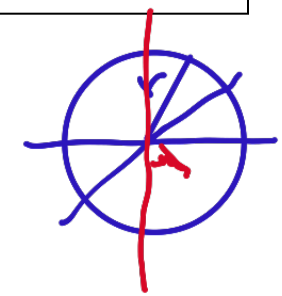
النظام الاحداثي القطبي

7-4- Polar coordinate system



$$\boxed{\vec{r} = r\,\vec{a}_r}$$





$$d\vec{r} = dr\,\vec{a}_r + r\,d\vec{a}_r \qquad \vec{a}_r = \cos\theta\,\vec{a}_x + \sin\theta\,\vec{a}_y$$

$$\frac{d\vec{a}_r}{d\theta} = -\sin\theta\,\vec{a}_x + \cos\theta\,\vec{a}_y = \vec{a}_\theta \quad \Rightarrow \quad d\vec{a}_r = d\theta\,\vec{a}_\theta$$

$$\boxed{d\vec{r} = dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta}$$

في الاحداثيات القطبيه : نفس الاحداثيات الزاويه لكننا نحذف Φ

8- Dot Notation

Dot Notation: $\dfrac{dx}{dt} = \dot{x}, \quad \dfrac{dy}{dt} = \dot{y}, \quad \dfrac{dz}{dt} = \dot{z}, \quad \dfrac{d^2 z}{dt^2} = \ddot{z}, \quad \dfrac{d\rho}{dt} = \dot{\rho}$

$$\frac{d\theta}{dt} = \dot{\theta}, \quad \frac{dr}{dt} = \dot{r}, \quad \frac{d\phi}{dt} = \dot{\phi}, \quad \vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

We will use this dot notation extensively. It means

dot اشتقاق بالنسبة ل t

differentiation with respect to time, *t*, only.

$y' = \dfrac{dy}{dx}$

$\dot{y} = \dfrac{dy}{dt}$

$$\frac{dy}{dx} = y' \neq \dot{y}$$

طرق التعبير عن مشتقة

y'

$\dfrac{dy}{dx}$

$\dot{y}$