1. Introduction to Differential Equations

1.1 Definitions

Definition 1.1.1 **Differential Equation (DE):** An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

تابع

مستقل

Example:

مثال على معادلة تفاضلية

$\frac{dy}{dx} + 2x = 0$ (y is dependent variable and x is independent variable)

DE (معادلة التفاضلية)

$$y' + 1 = x$$

المعادلة التفاضلية هي اي معادلة تحتوي على مشتقة لعامل تابع او اكثر والنسبة لعامل مستقل او اكثر

dep ⇒ تابع Indp ⇒ مستقل

$y = 3x + 4$ $x \Rightarrow$ Indp

$y \Rightarrow$ dep

$\frac{dy}{dx} \Rightarrow$ مشتقة y بالنسبة لـ x

$x \Rightarrow$ Indp

$y \Rightarrow$ dep

$\frac{dx}{dt} \Rightarrow$ مشتقة x بالنسبة لـ t

$t \Rightarrow$ Indp

$x \Rightarrow$ dep

مشتقة اولى مشتقة ثانية مشتقة ثالثة

y' $\frac{dy}{dx}$ y'' $\frac{d^2y}{dx^2}$ y''' $\frac{d^3y}{dx^3}$

تربيع $\left(\frac{dy}{dx}\right)^2$ $\left(\frac{dy}{dx}\right)^3$ تكعيب

DE PDE
ODE

انواع المعادلات التفاضلية

Types of Differential Equations: There are two types of differential equations:

1. Ordinary differential equations (ODE). معادلات تفاضلية اعتيادية
2. Partial differential equations (PDE). معادلات تفاضلية جزئية

معادلات تفاضلية اعتيادية

Ordinary differential equation (ODE): If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an ordinary differential equation (ODE).

معادلة تحتوي على مشتقات اعتيادية $\frac{dy}{dx}$
وفيها متغير مستقل واحد ويمكن
اكثر من متغير تابع
(نشتق بالنسبة لمتغير واحد)

Example:

1. $\frac{dy}{dx}+5y=e^x$
2. $\frac{d^2y}{dx^2}-\frac{dy}{dx}+6y=0$
3. $\frac{dx}{dt}+\frac{dy}{dt}=2x+y$

$$\frac{dy}{dx}+2x=4$$

$$\frac{dy}{dx}+\frac{du}{dx}=5$$

ليست معادلة تفاضلية اعتيادية $\frac{dy}{dx}+\frac{dy}{dr}=x$

معادلات تفاضلية جزئية

Partial differential equation (PDE): An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a partial differential equation (PDE).

$$F(x,y,z,z_x,z_y,z_{xx},z_{xy},z_{yy},\cdots)=0$$

معادلة تحتوي على مشتقات جزئية $\frac{\partial y}{\partial x}$
لمتغير تابع او اكثر بالنسبة
لمتغيرين مستقلين او اكثر

Example:

1. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$
2. $y_{tt}=c^2u_{xx}$ $\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$
3. $\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0$
4. $\frac{\partial^2 u}{\partial t^2}\frac{1}{k}\frac{\partial u}{\partial t}=0$

$$\frac{\partial y}{\partial x}+\frac{\partial y}{\partial t}=4$$

$$F=4y+5x+3 \qquad F(x,y,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial^2 f}{\partial x^2}$$

F_x مشتقة الدالة f بالنسبة لـ x

F_y " " " " y لـ

F_{xx} مشتقه الداله مرتين بالنسبه لـ x

F_{yy} y 〃 〃 〃 〃 〃

F_{xy} مشتقه الداله بالنسبه لـ y ثم بالنسبه لـ x

رتبه المعادله التفاضلية

Order of a DE: The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation.

Example:

رتبه 2

1. $\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3-4y=e^x$ Order $\to 2$
2. $y'-4y=x^2+1$ Order $\to 1$
3. $y'''+3(y')^2=\frac{1}{2}y$ Order $\to 3$

$\frac{d^3y}{dx^3}$ مشتقة ثالثة

$\left(\frac{dy}{dx}\right)^3$ اس

* رتبة المعادلة: هي اعلى مشتقة فيها

درجه المعادله التفاضليه

Degree of a DE: The degree of a D.E. is the degree of the highest derivative which occur in it, after the D.E. has been made free from radicals and fractions as for as the derivatives are concurred.

Example:

1. $\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3-4y=\sin x$ degree $\to 1$ order = 2, degree = 1
2. $\left(\frac{dy}{dx}\right)^4-4x=e^x$ degree $\to 4$ order = 1
3. $y'''+3(y')^2=\frac{1}{2}y$ degree $\to 1$ order = 3 degree = 1

الدرجه هي الاس لاعلى مشتقة

$\frac{dy}{dx}+3x=4$ رتبه = 1
درجة = 1

$\left(\frac{d^2y}{dx^2}\right)^5+3x=4$ رتبه = 2
درجه = 5

معادله تفاضليه خطيه

Linear differential equation: A linear differential equation of order n (with y dependent and x is independent), is a differential equation of the form

We say that the differential equation is said to be linear if

1. The dependent variable and all its derivatives are of the first degree.
2. Each coefficient depends only on the independent variable. Otherwise it is nonlinear.

① التابع (y) هو ومشتقاته من الدرجه الاولى (ليس له اس)

② كل المعاملات يجب ان تكون معتمده على X او رقم ثابت

$\boxed{\square}\ \frac{dy}{dx}$

$$\boxed{a_n(x)}\frac{d^3y}{dx^3} + \boxed{a_{n-1}(x)}\frac{d^2y}{dx^2} + \boxed{a_x}\frac{dy}{dx} + \boxed{a_x}\,y = F(x)$$

Example: هل هذه المعادلات خطيه او لا

1. $\frac{d^2y}{dx^2} + 5y^2 = 0$ nonlinear (power not 1)
2. $\frac{d^2y}{dx^2} - y = \sin y$ nonlinear (nonlinear function of y)
3. $(1-x)\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + y = e^x$ linear

▪ **Example 1.1**

List of Differentials Equations

No	Differentials Equations	Order رتبه	Degree درجه	Linearity خطي او لا
1	$x\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^4 + y = 0$	3	1	Nonlinear
2	$t^5y^{(4)} - t^3y'' + 6y = 0$	4	1	Linear
3	$\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$	2	1	Nonlinear
4	$(\sin x)y''' - (\cos x)y'' = 2$	3	1	Linear
5	$\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = \cos(x+y)$	2	2	Nonlinear
6	$(1-x)y'' - 4xy' + 5y = \cos x$	2	1	Linear

$y' \quad y'' \quad y''' \quad y^{(4)}$ مشتقه

$y \quad y^2 \quad y^3 \quad y^4$ اس

■ **Example 1.2** Determine weather the given first-order differential equation is linear or nonlinear and indicate dependent variable $(y^2-1)dx + xdy = 0$.

$\frac{dy}{dx}$ $\frac{dx}{dy}$

$$\frac{(y^2-1)\,\cancel{dx}}{\cancel{dx}} + x\frac{dy}{dx} = 0$$

نقسم جميع المعادله على dx

$$(y^2-1) + x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + (y^2-1) = 0$$

non linear
in case x
is dependant

$$(y^2-1)\frac{dx}{dy} + x\frac{\cancel{dy}}{\cancel{dy}} = 0$$

نقسم جميع المعادله على dy

$$(y^2-1)\frac{dx}{dy} + x = 0$$

linear.
in case x
is dependant

1.2 Types of Solution of Differential Equation

1.2.1 Solution of Differential Equation

هل y هي حل للمعادلة:

■ **Example 1.3** Verify that $y = -\frac{1}{3}e^x$ is a solution of the differential equation $y'' - 4y = e^x$

الحل

$y = -\frac{1}{3}e^x$

$y' = -\frac{1}{3}e^x$

$y'' = -\frac{1}{3}e^x$

$$y'' - 4y = e^x$$

$$\underbrace{\left(-\tfrac{1}{3}e^x\right) - 4\left(-\tfrac{1}{3}e^x\right)}_{LHS} = \underbrace{e^x}_{RHS}$$

$$-\frac{1}{3}e^x + \frac{4}{3}e^x = e^x$$

$$e^x\left(-\frac{1}{3} + \frac{4}{3}\right) = e^x$$

$$e^x\left(\frac{\cancel{3}}{\cancel{3}}\right) = e^x$$

$$e^x = e^x$$

$$LHS = RHS$$

$y = -\frac{1}{3}e^x$ is solution of the equation

مراجعة

$$\boxed{\frac{d}{dx}e^x = e^x} \qquad \boxed{\frac{d}{dx}e^{3x} = 3e^{3x}}$$

■ **Example 1.4** Verify that $\boxed{y = xe^x}$ is a solution of the differential equation Solution $y'' - 2y' + y = 0$.

$y'' - 2y' + y = 0$:حلاً للمعادلة $\boxed{y = xe^x}$ اثبت ان

$$\boxed{y = xe^x}$$

$$y' = xe^x + e^x = \boxed{e^x(x+1)}$$

$$y'' = e^x(1) + (x+1)e^x = e^x + xe^x + e^x$$

$$\boxed{y'' = e^x(2+x)}$$

نعوض في المعادلة:

$$y'' - 2y' + y = 0$$

$$e^x(2+x) - 2e^x(x+1) + xe^x = 0$$

$$\cancel{2e^x} + xe^x - 2xe^x - \cancel{2e^x} + xe^x = 0$$

$$2\cancel{xe^x} - 2\cancel{xe^x} = 0$$

$$0 = 0$$

$$LHS = RHS$$

■ **Example 1.5** Verify that $y = e^{2x} + xe^{2x}$ is a solution of the differential equation Solution $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

$$\boxed{y = e^{2x} + xe^{2x}}$$

$$y'' - 4y' + 4y = 0$$

$$y' = 2e^{2x} + 2xe^{2x} + e^{2x}$$

$$\boxed{y' = 3e^{2x} + 2xe^{2x}}$$

$$y'' = 3.2e^{2x} + 2x(2e^{2x}) + 2e^{2x}$$

$$y'' = \underline{6e^{2x}} + 4xe^{2x} + \underline{2e^{2x}}$$

$$\boxed{y'' = 8e^{2x} + 4xe^{2x}}$$

الان نعوض في المعادلة

$$y'' - 4y' + 4y = 0$$

$$8e^{2x} + 4xe^{2x} - 4(3e^{2x} + 2xe^{2x}) + 4(e^{2x} + xe^{2x}) = 0$$

$$\underline{8e^{2x}} + 4xe^{2x} - 12\underline{e^{2x}} - 8xe^{2x} + \underline{4e^{2x}} + 4xe^{2x} = 0$$

$$\cancel{e^{2x}}[\cancel{8} + \underline{4\cancel{x}} - \cancel{12} - \underline{8\cancel{x}} + \cancel{4} + \underline{4\cancel{x}}] = 0 / e^{2x}$$

$$0 = 0$$

$$LHS = RHS$$

Explicit Solution $y = \sqrt{x^2 - 25}$

حل صريح

Implicit solution $y^2 x = \frac{\sqrt{x+y}}{2x}$

حل ضمني

1.2.2 Explicit Solution

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an explicit solution.

> ▪ **Example 1.6** Verify that the indicate function $y = \phi(x)$ is an explicit solution solution of the given first-order differential equation $(y-x)y' = y - x + 8$; $y = x + 4\sqrt{x+2}$.

$$(y-x)y' = y - x + 8 \qquad \boxed{y = x + 4\sqrt{x+2}}$$

$$y' = 1 + \frac{2}{\sqrt{x+2}}$$

$$(y-x)\left(1 + \frac{2}{\sqrt{x+2}}\right) \overset{?}{=} y - x + 8$$

$$(y-x)(1) + \frac{2(y-x)}{\sqrt{x+2}} \overset{?}{=} y - x + 8$$

$$y - x + \frac{2(\cancel{x} + 4\cancel{\sqrt{x+2}} - \cancel{x})}{\cancel{\sqrt{x+2}}} \overset{?}{=} y - x + 8$$

$$y - x + 8 = y - x + 8$$

$$\boxed{y = x + 4\sqrt{x+2}}$$

$$\boxed{\begin{aligned} \frac{d}{dx}\sqrt{x} &= \frac{1}{2\sqrt{x}} \\ \frac{d}{dx}\sqrt{4x} &= \frac{4}{2\sqrt{4x}} \end{aligned}}$$

$$\boxed{\begin{aligned} &x + 2 \geq 0 \\ &x \geq -2 \quad [-2, \infty) \\ &\text{domain} \end{aligned}}$$

1.2.3 Implicit Solution

حل ضمني

A relation $G(x,y)=0$ is said to be an implicit solution of an ordinary differential equation on an interval I, provided that there exists at least one function Φ that satisfies the relation as well as the differential equation on I.

■ **Example 1.7** Verify that the indicate expression is an ~~explicit~~ implicit solution solution of the given first-order differential equation [redacted] $2xydx+(x^2-y)dy=0$; $\quad -2x^2y+y^2=1$.

ضمني

$$-2x^2y+y^2=1$$

اثبت ان المعادله هي حل للمعادله التفاضليه

$$2xy\,dx+(x^2-y)\,dy=0 \qquad -2x^2y+y^2=1$$

نشتق هذه المعادله اشتقاق ضمني

$$-2x^2y+y^2=1$$

$$-2x^2\frac{dy}{dx}+-4xy+2y\frac{dy}{dx}=0$$

ضرب كل المعادله ب dx

$$-2x^2dy-4xy\,dx+2y\,dy=0$$

نقسم المعادله على -2

$$+x^2dy+2xy\,dx-y\,dy=0$$

$$\left[(x^2-y)\,dy+2xy\,dx=0\right]$$

نعم يعتبر حل للمعادله

حل explicit solution

$$-2x^2y+y^2=1$$

$$y^2-2x^2y-1=0$$

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$y=\frac{--2x^2\pm\sqrt{4x^4+4}}{2}$$

الاشتقاق الضمني

$$x^2+y^2=3x$$

$$2x+2y\frac{dy}{dx}=3$$

$$y = +\frac{\cancel{2}x^2 \pm \cancel{2}\sqrt{x^4+1}}{\cancel{2}}$$

$$y = x^2 \pm \sqrt{x^4+1} \begin{cases} x^2+\sqrt{x^4+1} \\ x^2-\sqrt{x^4-1} \end{cases}$$

$(-\infty, \infty)$

1.3 Initial-Value Problem

بعد حل المعادلة كان الحل

$y = x + c$ رقم ثابت

$y(1) = 4$ IVP لحساب هذا الرقم الثابت

$\underbrace{1}_{x}$ $\underbrace{4}_{y}$

IVP : Initial Value نستخدم

لحساب الثوابت في الحل

■ **Example 1.8** If $y = \dfrac{1}{x^2+c}$ is a one-parameter family of solutions of the first order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(2) = \dfrac{1}{3}$ $\Rightarrow$ $y = \dfrac{1}{x^2-1}$
2. $y(-2) = \dfrac{1}{2}$ $\Rightarrow$ $y = \dfrac{1}{x^2-2}$

1) $y = \dfrac{1}{x^2+c}$ المطلوب حساب $\boxed{c}$

$y(2) = \dfrac{1}{3}$

2 لـ x، $\dfrac{1}{3}$ لـ y

$$\frac{1}{3} = \frac{1}{2^2+c}$$

$$\frac{1}{3} = \frac{1}{4+c}$$

$3 = 4 + c$ $\quad 3 - 4 = c$

$\boxed{c = -1}$

2) $y(-2) = \frac{1}{2}$

$x \qquad\qquad y$

$$y = \frac{1}{x^2 + c}$$

$$\frac{1}{2} = \frac{1}{4 + c}$$

$4 + c = 2$

$c = 2 - 4$

$\boxed{c = -2}$