Logic:

Propositional Logic (continue)

Acknowledgment: All course slides are either referenced to Rosen Book online presentations (with certain amendments) or are personally developed by the instructors.

Previously,

- A ***proposition*** is a statement that is either true or false.
  - A proposition has a ***truth value***. T و F
  - A truth value can only be true or false; it can not be **none** or **both**
  - A statement is **not** a proposition if it is: الجمله لا تكون شرطيه
    - A question, command, or contain unknown variable(s).
  - It can be assigned to a ***propositional variable***.

الجمل، الطرفيه ١ أو ٢ جمله

- New ***compound proposition*** can created as a result of applying ***logical connectives*** on one or more propositions.

– Negation	$\neg$	نفي	NOT	
– Conjunction	$\land$	و	AND	
– Disjunction	$\lor$	أو	OR	(inclusive or)
– Exclusive OR	$\oplus$		XOR	(both is not accepted)

Previously,

- To find the truth value of a given compound proposition, we create a truth table.
  - Start with a column for every propositional variable in the preposition.
  - #rows = $2^{\#variables}$

Negation: $\neg p$

p	¬p
T	F
F	T

- **Example:**
  - p : The earth is round.
  - $\neg p$:
    - It is *not the case that* the earth is round
    - The earth is *not* round.

Conjunction: $p \wedge q$

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**
  - p : I am at home.
  - q : It is raining.
  - $p \wedge q$:
  - I am at home and it is raining

Disjunction: $p \vee q$

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**
  - p : I am at home.
  - q : It is raining.
  - $p \vee q$:
  - I am at home or it is raining

Exclusive OR: $p \oplus q$

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

- **Example**
  - p : I am at home.
  - q : It is raining.
  - $p \oplus q$:
  - I am at home or it is raining but not both

Agenda

- Connectives
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth tables of compound propositions
- Logical equivalence
- Propositional satisfiability

Implication [illegible]

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- If p and q are propositions, then $p \to q$ is a ***conditional statement*** or ***implication*** which is read as "if p, then q".

- It has this truth table:

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \to q$ denotes "If I am at home then it is raining."
- In $p \to q$, p is the *hypothesis* and q is the *conclusion*.

فرضية النتيجة

Understanding Implication

مجرد

- In $p \rightarrow q$ there does not need to be any connection between the hypothesis or the conclusion. The “meaning” of $p \rightarrow q$ depends only on the truth values of p and q.

- These implications are perfectly fine, but would not be used in ordinary English.
  - If the clouds are made of cotton candy, then I have more money than Bill Gates.
  - if UQU is opened every Friday then 2 is a prime.

Understanding Implication (cont)

مكرر

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes." $T \rightarrow F$
  - "If you get 100% on the final, then you will get an A."

- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Understanding Implication (cont)

- Assume you have these propositions:
  - p : You exercise three times a week. — تتمرن 3 مرات في الاسبوع
  - r : You follow a low-carb diet. — تتبع حمية
  - q : You lose weight. — تخسر وزن

p	q	p → q	Explanation
T	T	T	It is accepted. If you exercise, then naturally you will lose weight.
T	F	F	It is rejected. Since one of the causes is true, the result must be fulfilled.
F	T	T	It is accepted. There are other causes to lose weight not just exercising.
F	F	T	It is accepted. If the you don't exercise, then you may not lose weight.

Different Ways of Expressing $p \rightarrow q$

طرق التعبير عن السببيه في اللغة

if p, then q

if p, q

q unless $\neg p$

q if p

q whenever p

q follows from p

p implies q

p only if q

q when p

p is sufficient for q

q is necessary for p

a necessary condition for p is q

a sufficient condition for q is p

معاكوس

Converse, Contrapositive, and Inverse

- From $p \rightarrow q$ we can form new conditional statements:

$q \rightarrow p$	is the **converse** of $p \rightarrow q$	$q \rightarrow p$ converse $p \rightarrow q$
$\neg q \rightarrow \neg p$	is the **contrapositive** of $p \rightarrow q$	$p \rightarrow q$ $\neg q \rightarrow \neg p$
$\neg p \rightarrow \neg q$	is the **inverse** of $p \rightarrow q$	$p \rightarrow q$ $\neg p \rightarrow \neg q$

$\neg p \rightarrow \neg q$ $p \rightarrow q$

Example

حصول المطر هو سبب كافي ليمنعني من الذهاب الى المدينة

Find the converse, inverse, and contrapositive of "*It is raining is a sufficient condition for me not going to town.*"

If its. raining then I'm not going to

p sufficient q

if p then q

- p = It is raining q = I'm not going to town

  converse: $q \rightarrow p$

  If I'm not going to town, then it is raining.

  $p \rightarrow q$ if $q \rightarrow p$ then

  inverse: $\neg p \rightarrow \neg q$

  If it is not raining, then I'm going to town.

  $p \rightarrow q$ $\neg p \rightarrow \neg q$

  contrapositive: $\neg q \rightarrow \neg p$

  If I'm going to town, then it is not raining.

  $p \rightarrow q$ $-q \rightarrow \neg p$

Biconditionals

اداة الشرط الثنائية

- If p and q are propositions, then we can form the ***biconditional statement***, $p \leftrightarrow q$, read as "p **if and only if** q."

$\leftrightarrow$

- It has the following truth table:

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

p if and only if q

- If p denotes "I am at home." and q denotes "It is raining." Then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditionals

التعبير عن
في اللغة

بدائل

- Some alternative ways "*p* if and only if *q*" is expressed in English:
  - *p* **is necessary and sufficient for** *q*
  - **if** *p* **then** *q* **, and conversely**

    الشرط وعكسه

  - *p* **iff** *q*

Truth Tables For Compound Propositions

Construct truth table

- Construction of a truth table:

الصف

- Rows

نضع جميع الاحتمالات للصواب والخطأ لكل الجمل

  - Need a row for every possible combination of values for the propositional variables.
  - The number of rows in the table with n variables = 2^n

3 جمل	جملتين	جملة 1
8	4	2

العمود

- Columns

$p \land q$

  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.
    - This includes the propositional variables

جملة واحدة فقط صفين

p
T
F

جملتين 4 صفوف

p	q
T	T
T	F
F	T
F	F

3 جمل 8 صفوف

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Example Truth Table

انشئ جدول صدق

- Construct a truth table for:

$$[p \vee q] \rightarrow [\neg r]$$

p	q	r	① $p \vee q$	② $\neg r$	① → ②
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

Example Truth Table

- Construct a truth table for:

$$p \vee q \rightarrow \neg r$$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Precedence of Logical Operators

اولوية العمليات المنطقية

Operator	Precedence
$\neg$	1 النفي
$\wedge$ $\vee$	2 و 3 او
$\rightarrow$ $\leftrightarrow$	4 السببيه 5 السببية الثنائيه

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

الاقواس تسبق الجميع
اذا تساوت العمليات نبدأ من اليمين

If the intended meaning is $p \vee (q \rightarrow \neg r)$, then parentheses must be used.

$(p \wedge q) \vee \sim q$

p	q	p∧q	~q	(p∧q) V~q
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	T	T

امثلة اضافية

p	q	p∧q	¬q	(p∧q)∨~q
T	T	T	F	T
T	F	F	T	T
F	T	F	F	F
F	F	F	T	T

$(p \wedge q) \vee (p \vee q)$

Answer

p	q	p∧q	p∨q	(p∧q)∨(p∨q)
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

Agenda

- Connectives
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth tables for compound propositions
- Logical equivalence
- Propositional satisfiability

تصنيف الجمل الطرفيه المركبة

Classification of compound propositions

جملة دائما تكون صحيحة

tautology

- A **tautology** is a proposition which is always true.
  - Example: $\boxed{p \vee \neg p}$

نقوم بعمل جدول صدق وإذا كان العمود الأخير (T)

- A **contradiction** is a proposition which is always false.
  - Example: $\boxed{p \wedge \neg p}$

دائما تكون خاطئة

- A **contingency** is a proposition which is neither a tautology nor a contradiction, such as p

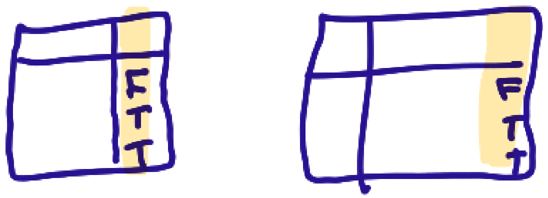
الجملة الشرطية دائما صواب و لا خطأ

tautology contradiction

p	¬p	p ∨¬p	p ∧¬p
T	F	T	F
F	T	T	F

تكافؤ منطقي

✓ $p \equiv q$ ✓



Logically Equivalent

- Two compound propositions p and q are **logically equivalent if** $p \leftrightarrow q$ **is a tautology**.

- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

- Two compound propositions p and q are ***equivalent*** if and only if the columns in a truth table giving their truth values agree.

Equivalent Propositions

$\neg q \rightarrow \neg p \equiv (p \rightarrow q)$ استخدم جدول الصدق لاثبات ان

- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.
  - Conditional: $\mathbf{p} \rightarrow \mathbf{q}$ Contrapositive: $\neg\mathbf{q} \rightarrow \neg\mathbf{p}$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

$$(p \rightarrow q) \equiv \neg q \rightarrow \neg p$$

Equivalent Propositions

- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.
  - Conditional: $p \rightarrow q$ Contrapositive: $\neg q \rightarrow \neg p$

p	q	¬ p	¬ q	p → q	¬q → ¬ p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$$

$\equiv$

Equivalent Propositions

- **Example**: Show using a truth table that $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$$\neg p \vee q \equiv p \rightarrow q$$

Equivalent Propositions

- **Example**: Show using a truth table that $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	¬p	¬p ∨ q	p→ q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$$\neg p \rightarrow q \Leftrightarrow p \rightarrow q$$

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

$q \to p$

$p \to q$

$\neg p \to \neg q$

المطلوب اثبات ان $(q \to p)$ و $(\neg p \to \neg q)$ لا تتطابق مع $(p \to q)$

p	q	$\neg p$	$\neg q$	$q \to p$	$\neg p \to \neg q$	$p \to q$
T	T	F	F	T	T	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Converse: $q \rightarrow p$ inverse: $\neg p \rightarrow \neg q$

Implication: $p \rightarrow q$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

De Morgan's Laws

قوانين دي مورجان

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

$$\boxed{\neg(p \lor q) \equiv \neg p \land \neg q}$$

اثبت قوانين دي مورجان

Augustus De Morgan
1806-1871

This truth table shows that De Morgan's Second Law holds.

p	q	¬p	¬q	(p∨q)	¬(p∨q)	¬p∧¬q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

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Key Logical Equivalences

حفظ

- عنصر محايد
  Identity Laws: $p \land T \equiv p$, $p \lor F \equiv p$
- السيادة
  Domination Laws: $p \lor T \equiv T$, $p \land F \equiv F$
- جبر
  Idempotent laws: $p \land p \equiv p$, $p \lor p \equiv p$
- النفي المزدوج
  Double Negation Law: $\neg(\neg p) \equiv p$
- Negation Laws: $p \lor \neg p \equiv T$, $p \land \neg p \equiv F$

Key Logical Equivalences (*cont*)

التبديلي

- Commutative Laws: $p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$

$S \vee (T \wedge W)$
$= (S \vee T) \wedge (S \vee W)$

- Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  $(p \vee q) \vee r \equiv p \vee (q \vee r)$

التجميع

يمكن توزيع $\vee$ على قوس $\wedge$ والعكس

- Distributive Laws: $(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$
  $(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$

التوزيع

الامتصاص

- Absorption Laws: $(p \vee (p \wedge q)) \equiv p$, $(p \wedge (p \vee q)) \equiv p$

$(p \vee p) \wedge (p \vee q)$
$p \wedge (p \vee q)$

More Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.
١ الزجبة درس
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- We can show that two expressions are logically equivalent by developing a series of logically equivalent statements.

- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$A \equiv A_1$
$A_1 \equiv A_2$
$A_2 \equiv A_3 \equiv B$

$$A \equiv A_1$$
$$\vdots$$
$$A_n \equiv B$$

- Keep in mind that whenever a proposition (represented by a propositional variable) occurs in the equivalences listed earlier, it may be replaced by an arbitrarily complex compound proposition.

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge [\neg(\neg p \wedge q)] && \text{De Morgan} \\
&\equiv \neg p \wedge (\neg\neg p \vee \neg q) && \text{De Morgan} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{double negation} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{Distribution} \\
&\equiv F \vee (\neg p \wedge \neg q) && (\neg p \wedge p = F) \\
&\equiv \neg p \wedge \neg q && \text{Identity}
\end{aligned}
$$

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

Solution:

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv F \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv F \\
&\equiv (\neg p \wedge \neg q) \vee F && \text{by the commutative law for disjunction} \\
&\equiv (\neg p \wedge \neg q) && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology. T

Solution:

$$
\begin{array}{rcll}
(p \wedge q) \rightarrow (p \vee q) & \equiv & \neg(p \wedge q) \vee (p \vee q) & \text{by truth table for } \rightarrow \\
& \equiv & (\neg p \vee \neg q) \vee (p \vee q) & \text{by the first De Morgan law} \\
& \equiv & (\neg p \vee p) \vee (\neg q \vee q) & \text{by associative and} \\
& & & \text{commutative laws} \\
& & & \text{laws for disjunction} \\
& \equiv & T \vee T & \text{by truth tables} \\
& \equiv & T & \text{by the domination law}
\end{array}
$$

Today's Agenda

- Connectives
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth tables for compound propositions
- Logical equivalence
- Propositional satisfiability

Propositional Satisfiability

- A compound proposition is ***satisfiable*** if there is an assignment of truth values to its variables that make it true.

اذا كانت احدى قيم الصف للجملة T تسمى satisfiable

- A compound proposition is ***unsatisfiable*** if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

لايوجد اي قيمه تؤدي الى T

- When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable; such an assignment is called a ***solution***.

هل الجملة هي جميع الاحتمالات التي تعطي T

Questions on Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$p \to T \quad q \to T \quad r \to T$

$(T \vee F) \wedge (T \vee F) \wedge (T \vee F)$

$T \wedge T \wedge T = T$

في حاله كل الجمل تعطي $T = T$

- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

  Solution: Satisfiable. Assign **T** to p, q, and r.

$p = T \quad q = T \quad r = F$

- $(\overset{T}{p} \vee q \vee r) \wedge (\neg p \vee \overset{T}{\neg q} \vee \neg r)$

  Solution: Satisfiable. Assign **T** to p and $\boldsymbol{F}$ to q.

- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

  Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

Example: Determine the satisfiability of the following compound propositions:

- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$

  Solution: Satisfiable. Assign **T** to p, q, and r.

p	q	r	¬q	p∨¬q	¬r	q∨¬r	¬p	r∨¬p	(q∨¬r)∧(r∨¬p)	(p∨¬q)∧(q∨¬r)∧(r∨¬p)
T	T	T	F	T	F	T	F	T	T	T
T	T	F	F	T	T	T	F	F	F	F
T	F	T	T	T	F	F	F	T	F	F
T	F	F	T	T	T	T	F	F	F	F
F	T	T	F	F	F	T	T	T	T	F
F	T	F	F	F	T	T	T	T	T	F
F	F	T	T	T	F	F	T	T	F	F
F	F	F	T	T	T	T	T	T	T	T

- $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

  Solution: Not satisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.

p	q	r	¬q	p∨¬q	¬r	q∨¬r	¬p	r∨¬p	q∨r	p∨q∨r	¬q∨¬r	¬p∨¬q∨¬r	(p∨q∨r)∧(¬p∨¬q∨¬r)	(r∨¬p)∧(p∨q∨r)∧(¬p∨¬q∨¬r)	(q∨¬r)∧(r∨¬p)∧(p∨q∨r)∧(¬p∨¬q∨¬r)	(p∨¬q)∧(q∨¬r)∧(r∨¬p)∧(p∨q∨r)∧(¬p∨¬q∨¬r)
T	T	T	F	T	F	T	F	T	T	T	F	F	F	F	F	F
T	T	F	F	T	T	T	F	F	T	T	T	T	T	F	F	F
T	F	T	T	T	F	F	F	T	T	T	T	T	T	T	F	F
T	F	F	T	T	T	T	F	F	F	T	T	T	T	F	F	F
F	T	T	F	F	F	T	T	T	T	T	F	T	T	T	T	F
F	T	F	F	F	T	T	T	T	T	T	T	T	T	T	T	F
F	F	T	T	T	F	F	T	T	T	T	T	T	T	T	F	F
F	F	F	T	T	T	T	T	T	F	F	T	T	F	F	F	F

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What is Next?

- **Read** sections 1.1.3, 1.1.4, 1.3.1 to 1.3.5 of Rosen’s book.
- **Practice and solve** the practice sheet in the Blackboard (all exercises).