CHAPTER 1

The Wave Function

1.1 The Schrodinger equation

$x(t)$

$v = \frac{dx}{dt}$

$a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$

$F = ma$

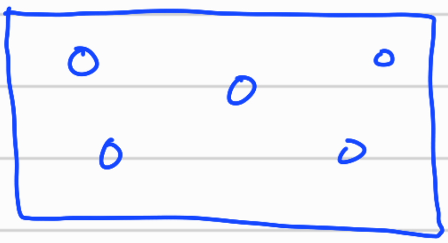
$T = \frac{1}{2} m v^2$

$F = -\frac{\partial V}{\partial x}$

$U = Fx$

باستخدام قوانين الميكانيكا الكلاسيكية من السهل التعرف على موقع وحركة وتسارع الجسم

✓ اذا اردنا معرفة موقع الالكترون نتعامل مع الالكترون كانه موجة



$\Psi(x,t)$

wave function

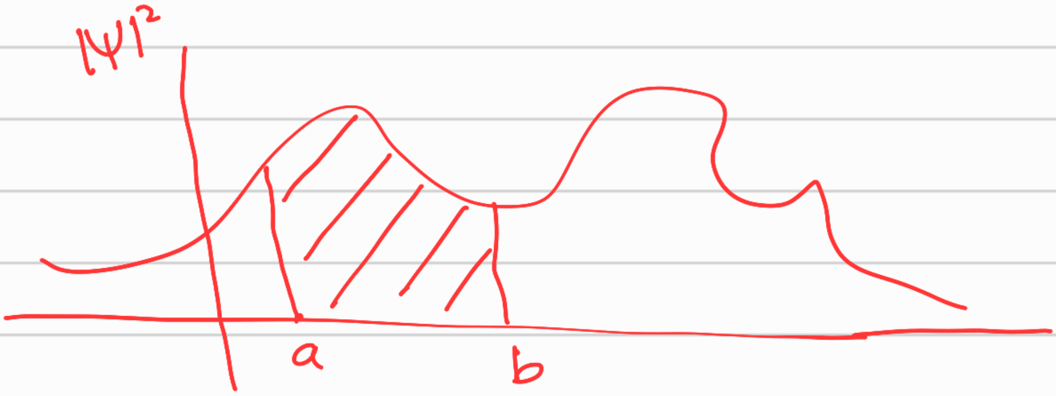
Schrodinger equation

$$\underbrace{i\hbar \frac{\partial \psi}{\partial t}}_{\text{Total energy}} = \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}}_{\text{KE طاقة حركية}} + \underbrace{V\psi}_{\text{طاقة وضع PE}}$$

$i = \sqrt{-1}$ $\qquad \hbar = \frac{h}{2\pi} = \frac{6.6 \times 10^{-34}}{2\pi}$

$\hbar = 1.0545 \times 10^{-34} \, Js$

1.2 the statistical Interpetation



$$\int_a^b |\psi|^2 \, dx$$

التعبير الاحصائي عن
احتمالية وجود الجسم
في مكان ما في المنطقة من
a → b

$\int_a^b |\psi(x,t)|^2 \, dx$ = the Probability of finding the particle betweon a and b

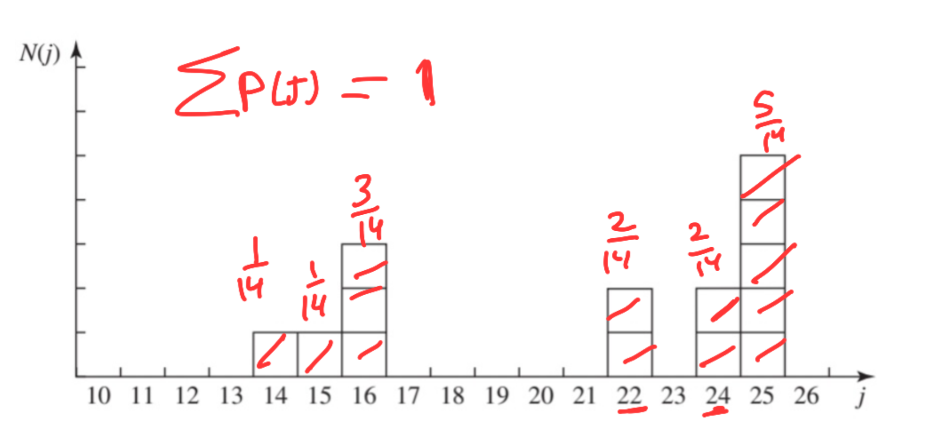
$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \, dx = 1$$

the Particle is at point c

1.3 Probability

* Discrete variables

$N(14) = 1$
$N(15) = 1$
$N(16) = 3$
$N(22) = 2$
$N(24) = 2$
$N(25) = 5$



Question 1 If you selected one individual at random from this group, what is the **probability** that this person's age would be 15?

$$P(15) = \frac{N(15)}{N} = \frac{1}{14}$$

$P(j)$ احتمالية عمر معين

N العدد الكلي

$N(j)$ عدد أفراد عمر معين

$$P(j) = \frac{N(j)}{N}$$

Question 2 What is the **most probable** age?

25

Question 3 What is the **median** age?

الفئة اللي تأتي بالوسط

23

Question 4 What is the **average** (or **mean**) age?

$$\langle j \rangle = \frac{j\,N(j)}{N} = \sum j\,P(j)$$

$$\langle j \rangle = \frac{14(1) + 15(1) + 16(3) + 22(2) + 24(2) + 25(5)}{14}$$

$$\langle j \rangle = 21$$

$$14 \times \frac{1}{14} + 15 \times \frac{1}{14} + 16 \times \frac{3}{14} + 22 \times \frac{2}{14} + 24 \times \frac{2}{14} + 25 \times \frac{5}{14}$$

j $P(j)$

$$= 21$$

$$\langle j \rangle^2 = 21^2 = 441$$

Question 5 What is the average of the *squares* of the ages?

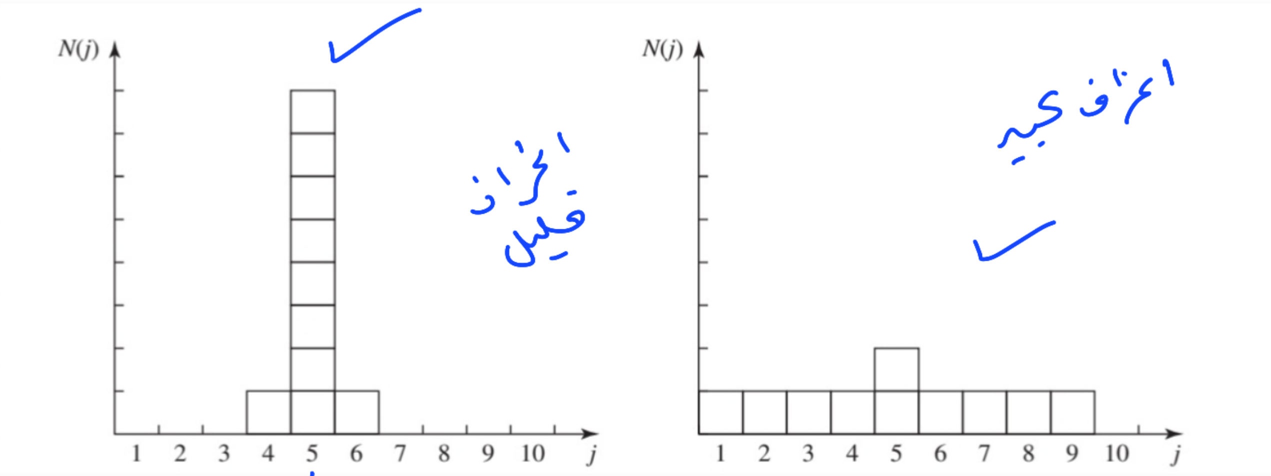
$$\langle j^2 \rangle = \sum j^2\,P(j)$$

$$\langle f(j) \rangle = \sum f(j)\,P(j)$$

$$= 14^2 \times \frac{1}{14} + 15^2 \times \frac{1}{14} + 16^2 \times \frac{3}{14} + 22^2 \times \frac{2}{14} + 24^2 \times \frac{2}{14} + 25^2 \times \frac{5}{14}$$

$$\langle j^2 \rangle = 459.6$$

$$\langle j \rangle^2 \neq \langle j^2 \rangle$$



انحراف قليل

انحراف كبير

mean
median = 5

mean
median = 5

standard deviation الانحراف المعياري

$$\sigma^2$$

تباين $\sigma \rightarrow$ variance

$$\boxed{\begin{aligned} \sigma^2 &= \langle j^2 \rangle - \langle j \rangle^2 \\ \sigma &= \sqrt{\langle j^2 \rangle - \langle j \rangle^2} \end{aligned}}$$

1.3.2 Continuous variables

Probability $= \rho(x)\,\underline{dx}$

احتماليه اختيار عدد في فتره معينه dx

Probability density $a \longrightarrow b$

$$P_{ab} = \int_a^b \rho(x)\,dx$$

$$\int_{-\infty}^{\infty} \rho(x)\,dx = 1$$

$$\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\,\rho(x)\,dx$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

حل تمارين كم (1) شابتر (1)

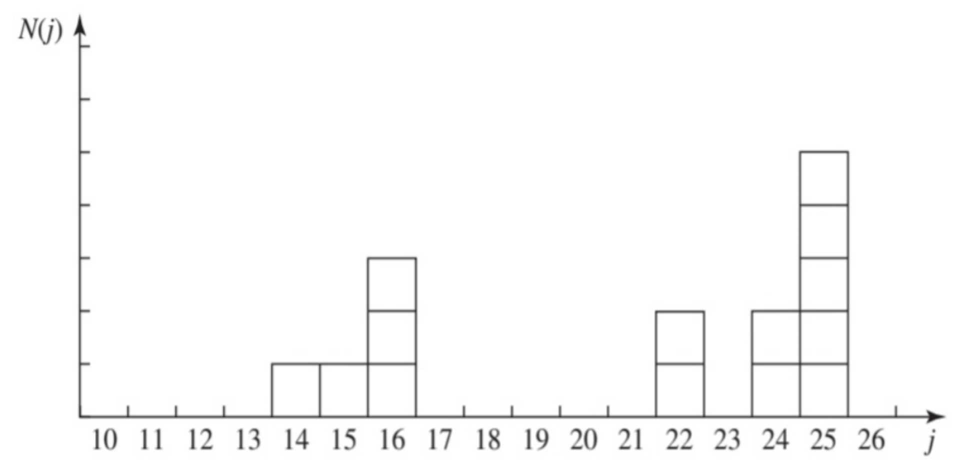
1.1 + 1.2 + 1.3

Problem 1.1 For the distribution of ages in the example in Section 1.3.1:

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$.

(b) Determine Δj for each j, and use Equation 1.11 to compute the standard deviation.

(c) Use your results in (a) and (b) to check Equation 1.12.



j	$N(j)$
14	1
15	1
16	3
22	2
24	2
25	5
	14

a) $\langle j^2 \rangle$ $\quad \langle j \rangle^2$

$$\langle j \rangle = \sum j P(j) = \sum \frac{j N(j)}{N}$$

$$\langle j \rangle = \frac{14(1)+15(1)+16(3)+22(2)+24(2)+25(5)}{14}$$

$$= 21$$

$$\langle j \rangle^2 = 21^2 = 441$$

$$\langle j^2 \rangle = \sum j^2 P(j) = \sum j^2 \frac{N(j)}{N}$$

$$\langle j^2 \rangle = \frac{14^2(1)+15^2(1)+16^2(3)+22^2(2)+24^2(2)+25^2(5)}{14}$$

$$\langle j^2 \rangle = 459.57$$

b)

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2}$$

(1.11) $$\sigma^2 = \langle \Delta j^2 \rangle = \frac{\sum (\Delta j)^2 N(j)}{N}$$

j	$\Delta j = j - \langle j \rangle$	$N(j)$
14	$14-21 = -7$	1
15	$15-21 = -6$	1
16	$16-21 = -5$	3
22	$22-21 = 1$	2
24	$24-21 = 3$	2
25	$25-21 = 4$	5

$$\sigma^2 = \frac{\sum \Delta j^2 \, N(j)}{N}$$

$$= \frac{(-7)^2 1 + (-6^2)1 + (-5)^2(3) + 1(2) + 3^2(2) + 4^2(5)}{14}$$

$\sigma^2 = 18.571$ $\sigma = 4.309$

c)

$$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2$$

$$= 459.571 - 441$$

$$\sigma^2 = 18.571$$

$$\sigma = 4.309$$

Problem 1.2 σ^2

(a) Find the standard deviation of the distribution in Example 1.2.

(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from the average?

$$P(x) = \frac{1}{2\sqrt{hx}} \quad (0 \le x \le h)$$

a)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot P(x)\, dx$$

$$\langle x \rangle = \int_0^h x \frac{1}{2\sqrt{hx}} = \int_0^h \frac{x^1}{\sqrt{x}} \frac{1}{2\sqrt{h}}$$

$$\langle x \rangle = \int_0^h \frac{x^{1/2}}{2\sqrt{h}}\, dx = \frac{1}{2\sqrt{h}} \int_0^h x^{1/2}\, dx$$

$$\langle x \rangle = \frac{1}{2\sqrt{h}} \left[\frac{x^{3/2}}{3/2} \right]_0^h$$

$$\langle x \rangle = \frac{1}{\cancel{2}\sqrt{h}} \frac{\cancel{2}}{3} x^{3/2} \Big]_0^h$$

$$\langle x \rangle = \frac{1}{3\sqrt{h}} \left[h^{3/2} - 0^{3/2} \right]$$

$$\langle x \rangle = \frac{1}{3} \frac{h^{3/2}}{h^{1/2}} = \frac{h}{3}$$

$$\langle x \rangle = \frac{h}{3}$$

$$\langle x \rangle^2 = \frac{h^2}{9}$$

$$\langle x^2 \rangle = \int_0^h x^2 \cdot \rho(x)$$

$$= \int_0^h x^2 \frac{1}{2\sqrt{hx}}$$

$$= \frac{1}{2\sqrt{h}} \int_0^h \frac{x^2}{x^{1/2}} = \frac{1}{2\sqrt{h}} \int_0^h x^{3/2}\,dx$$

$$= \frac{1}{2\sqrt{h}} \left.\frac{x^{5/2}}{5/2}\right]_0^h = \frac{1}{\cancel{2}\sqrt{h}} \frac{\cancel{2}}{5} \left[h^{5/2} - 0\right]$$

$$= \frac{1}{5\,h^{1/2}}\, h^{5/2} = \frac{h^2}{5}$$

$$\langle x^2 \rangle = \frac{h^2}{5}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{h^2}{5} - \frac{h^2}{9}$$

✓ $$\sigma^2 = h^2\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{4h^2}{45}$$

✓ $$\sigma = \sqrt{\frac{4}{45}}\, h$$

Problem 1.3 Consider the **gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are positive real constants. (The necessary integrals are inside the back cover.)

(a) Use Equation 1.16 to determine A.

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

(c) Sketch the graph of $\rho(x)$.

$$\int_{-\infty}^{+\infty} P(x)\, dx = 1$$

$$\rho(x) = A e^{-\lambda(x-a)^2}$$

$$1 = \int_{-\infty}^{\infty} P(x)\, dx = A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\, dx$$

بالتعويض مكان

$$u = x - a$$

$$du = dx$$

$$1 = A\int_{-\infty}^{\infty} e^{-\lambda u^2}\, du$$

$$\int_{0}^{\infty} e^{-au^2}\, du = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$1 = 2A\int_{0}^{\infty} e^{-\lambda u^2}\, du$$

$$1 = \cancel{2}A\left[\frac{1}{\cancel{2}}\sqrt{\frac{\pi}{\lambda}}\right]$$

$$1 = A\frac{\sqrt{\pi}}{\sqrt{\lambda}}$$

$$A = \frac{\sqrt{\lambda}}{\sqrt{\pi}} = \sqrt{\frac{\lambda}{\pi}}$$

b) $$\langle x \rangle = \int_{-\infty}^{+\infty} x \, P(x) \, dx$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \sqrt{\frac{\lambda}{\pi}} \, e^{-\lambda (x-a)^2} \, dx$$

$u = x - a \qquad x = u + a$ بتعويض

$du = dx$

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{+\infty} \underline{x} \, e^{-\lambda u^2} \, du$$

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (u + a) \, e^{-\lambda u^2} \, du$$

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \left[\underbrace{\int_{-\infty}^{\infty} u \, e^{-\lambda u^2} \, du}_{\text{odd function}} + a \int_{-\infty}^{\infty} e^{-\lambda u^2} \, du \right]$$

odd function

= zero

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \left[0 + 2a \int_{0}^{\infty} e^{-\lambda u^2} \, du \right]$$

$$\langle x \rangle = \cancel{\sqrt{\frac{\lambda}{\pi}}} \left[\cancel{2} a \, \frac{1}{\cancel{2}} \cancel{\sqrt{\frac{\pi}{\lambda}}} \right]$$

$$\langle x \rangle = a$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \, p(x) \, dx$$

$$= \int x^2 \sqrt{\frac{\lambda}{\pi}} \, e^{-\lambda (x-a)^2} \, dx$$

$u = x - a$

$du = dx$

$x = u + a$

$(u+a)^2 = \underline{u^2} + \underline{2au} + \underline{a^2}$

$$= \sqrt{\frac{\lambda}{\pi}} \int (u+a)^2 \, e^{-\lambda u^2} \, du$$

فك التربيع

$$= \sqrt{\frac{\lambda}{\pi}} \left[\int_{-\infty}^{+\infty} u^2 e^{-\lambda u^2} \underbrace{+ 2a \int u e^{-\lambda u^2}}_{\text{zero}} + \underbrace{a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2} du} \right]$$

$$= \sqrt{\frac{\lambda}{\pi}} \left[2 \int_{0}^{\infty} u^2 e^{-\lambda u^2} \, du + 2a^2 \int_{0}^{\infty} e^{-\lambda u^2} \, du \right]$$

$$= \cancel{\sqrt{\frac{\lambda}{\pi}}} \left[\cancel{2} \, \frac{1}{\underset{2}{\cancel{4}}\lambda} \cancel{\sqrt{\frac{\pi}{\lambda}}} + \cancel{2} a^2 \frac{1}{\cancel{2}} \cancel{\sqrt{\frac{\pi}{\lambda}}} \right]$$

$$\langle x^2 \rangle = \frac{1}{2\lambda} + a^2$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma = \sqrt{\frac{1}{2\lambda} + \cancel{a^2} - \cancel{a^2}}$$

$$\sigma = \sqrt{\frac{1}{2\lambda}}$$

1.4 Normalization

ψ wave function

1- ψ يجب ان تحقق معادلة شرودنغر

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial t^2} - \frac{i}{\hbar} V \psi$$

ψ ψ^*

i $-i$

$$\frac{\partial \psi^*}{\partial t} = \frac{-i\hbar}{2m}\frac{\partial^2 \psi}{\partial t^2} + \frac{i}{\hbar} V \psi$$

2- $\int |\psi^2| \, dx$ probability density

$$\int_{-\infty}^{\infty} |\psi^2| \, dx = 1$$

Normalized $\int_{-\infty}^{\infty} |\psi^2| \, dx = \int_{-\infty}^{\infty} \psi \psi^* \, dx = 1$

ψ non-normalized $\int_{-\infty}^{\infty} |\psi^2| \neq 1$

$$\int_{-\infty}^{\infty} A\psi = 1$$

Normalization : نحسب الثابت A الذي يجعل $\int_{-\infty}^{\infty} \psi \psi^* =$

$$\int_{-\infty}^{\infty} |\psi|^2 = 1 \qquad \frac{d}{dt}\int_{-\infty}^{\infty} |\psi|^2 = 0$$

Problem 1.4 At time $t = 0$ a particle is represented by the wave function

$$\Psi(x, 0) = \begin{cases} A(x/a), & 0 \le x \le a, \\ A(b-x)/(b-a), & a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are (positive) constants.

(a) Normalize Ψ (that is, find A, in terms of a and b).

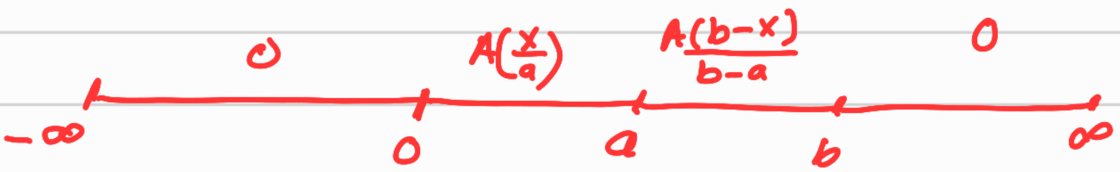
(b) Sketch $\Psi(x, 0)$, as a function of x.

(c) Where is the particle most likely to be found, at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

(e) What is the expectation value of x?

القيمة المتوقعة $\langle x \rangle = \int x \psi^2 dx$



a) $$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$= \int_{-\infty}^{0} |\psi|^2 dx + \int_{0}^{a} |\psi|^2 dx + \int_{a}^{b} |\psi|^2 dx + \int_{b}^{\infty} |\psi^2| dx = 1$$

$$= \cancel{\int_{-\infty}^{0} 0} + \int_{0}^{a} A^2 \frac{x^2}{a^2} dx + \int_{a}^{b} \frac{A^2 (b-x)^2}{(b-a)^2} dx + \cancel{\int_{b}^{\infty} 0} = 1$$

$$= \frac{A^2}{a^2} \int_{0}^{a} x^2 dx + \frac{A^2}{(b-a)^2} \int (b-x)^2 dx = 1$$

$$1 = \frac{A^2}{a^2}\left[\frac{x^3}{3}\right]_0^a + \frac{A^2}{(b-a)^2}\left[\frac{(b-x)^3}{-3}\right]_a^b$$

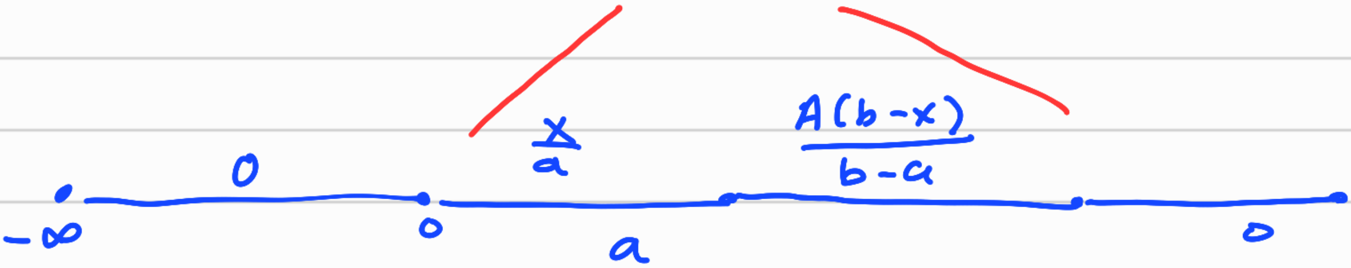
$$1 = \frac{A^2}{a^2}\left[\frac{a^3}{3}\right] + \frac{A^2}{-3(b-a)^2}\left[\cancel{(b-b)^3} - (b-a)^3\right]$$

$$= \frac{A^2}{\cancel{a^2}}\,\frac{\overset{a}{\cancel{a^3}}}{3} + \frac{A^2}{3\cancel{(b-a)^2}}\,\overset{(b-a)}{\cancel{(b-a)^3}} = 1$$

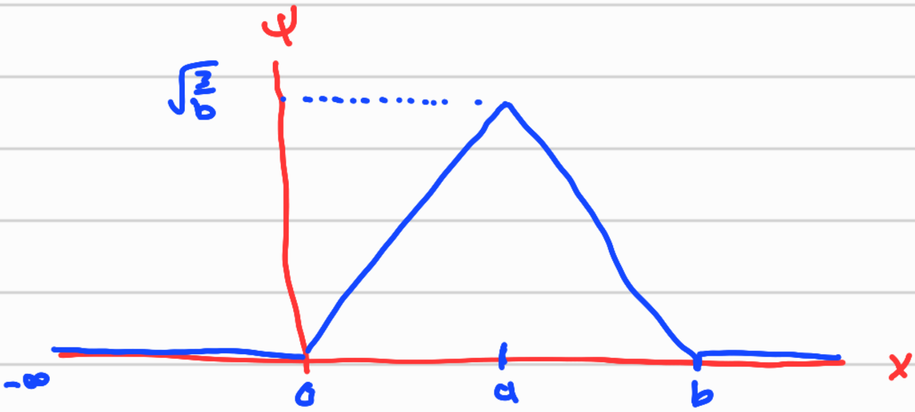
$$1 = \frac{A^2\cancel{a}}{\cancel{3}} + \frac{A^2 b}{3} - \frac{A^2\cancel{a}}{\cancel{3}}$$

$$1 = \frac{A^2 b}{3}$$

$$\frac{3}{b} = A^2 \qquad A = \sqrt{\frac{3}{b}}$$



b)



c) at $x = a$

d) $$P = \int_{-\infty}^{a} |\psi(x,0)|^2 \, dx$$

$$P = \cancel{\int_{-\infty}^{0} 0 \, dx} + \int_{0}^{a} A^2 \frac{x^2}{a^2} \, dx$$

$$P = \int_{0}^{a} \sqrt{\frac{3}{b}}^{\,2} \frac{x^2}{a^2}$$

$$P = \frac{3}{ba^2} \int_{0}^{a} x^2 = \frac{3}{ba^2} \left[\frac{x^3}{3} \right]_0^a$$

$$= \frac{3}{ba^2} \, \frac{a^3}{3} = \frac{a}{b}$$

$$P = \frac{a}{b}$$

$\Rightarrow$ in case $a = b$ $\quad P = \frac{a}{b} = \frac{a}{a} = 1$

in case $b = 2a$ $\quad P = \frac{a}{b} = \frac{a}{2a} = \frac{1}{2}$

$$\begin{cases} a = b & P = 1 \\ b = 2a & P = 1/2 \end{cases}$$

e)

$$\langle x \rangle = \int_{-\infty}^{\infty} x\,|\psi|^2\,dx \qquad A = \sqrt{\tfrac{3}{b}}$$

$$= \int_0^a x\,\frac{A^2 x^2}{a^2}\,dx + \int_a^b \frac{x A^2 (b-x)^2}{(b-a)^2}\,dx$$

$$= \frac{3}{ba^2}\int_0^a x^3\,dx + \frac{3}{b(b-a)^2}\int_a^b \underbrace{x}_{u}\,\underbrace{(b-x)^2\,dx}_{dv}$$

تكامل بالاجزاء

$$\frac{3}{ba^2}\left.\frac{x^4}{4}\right]_0^a + \frac{3}{b(b-a)^2}\left[\frac{x(b-x)^3}{-3} - \int \frac{(b-x)^3}{-3}\right]$$

$u = x \qquad dv = (b-x)^2$

$du = dx \qquad v = \dfrac{(b-x)^3}{-3}$

$uv - \int v\,du$

$$\frac{3a^4}{4ba^2} + \frac{3}{b(b-a)^2}\left[\frac{a(b-a)^3}{-3} - \frac{(b-x)^4}{12}\right]_a^b$$

$$\frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\left[\frac{a(b-a)^3}{-3} + \frac{(b-a)^4}{12}\right]$$

$$\frac{3a^2}{4b} - \frac{3a(b-a)^3}{3b(b-a)^2} + \frac{3(b-a)^4}{12b(b-a)^2}$$

$$\frac{3a^2}{4b} + \frac{a(b-a)}{b} + \frac{(b-a)^2}{4b}$$

$$\frac{3\cancel{a^2}}{\cancel{4}b} + \frac{4ab}{4b} - \frac{4\cancel{a^2}}{\cancel{4b}} + \frac{b^2}{4b}\left(-\frac{2ab}{4b} + \frac{\cancel{a^2}}{\cancel{4b}}\right)$$

$$= \frac{2ab}{4b} + \frac{b^2}{4b}$$

1.5 Momentum

$$\vec{p} = mv = m\frac{dx}{dt}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\,|\psi(x,t)|^2\,dx$$

$$\langle v \rangle = \frac{d\langle x \rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} x\,\psi^2\,dx$$

$$\langle v \rangle = \frac{d\langle x \rangle}{dt} = \frac{-i\hbar}{m}\int \psi^* \frac{\partial \psi}{\partial x}\,dx$$

$$p = m\frac{d\langle x \rangle}{dt} = \cancel{m}\left(\frac{-\hbar}{2\cancel{m}}\int \psi^* \frac{\partial \psi}{\partial x}\,dx\right)$$

$$P = -i\hbar \int \psi^* \frac{\partial \psi}{\partial x} \, dx$$

شكل آخر لمعادلة القيمة المتوقعة للموقع والزخم

$$\langle x \rangle = \int \psi^* [x] \, \psi \, dx$$

operator

$$\langle p \rangle = \int \psi^* \left[-i\hbar \frac{\partial}{\partial x} \right] \psi \, dx$$

operator

operator

$$\langle x \rangle \Longrightarrow [x]$$

$$\langle p \rangle \Longrightarrow \left[-i\hbar \frac{\partial}{\partial x} \right]$$

$$\langle Q \rangle \Longrightarrow [Q]$$

Kinetic energy T

$$T = \frac{1}{2} m v^2 = \frac{p^2}{2m}$$

$p^2 = m^2 v^2 \quad v^2 = \frac{p^2}{m^2}$

$T = \frac{1}{2} m v^2$

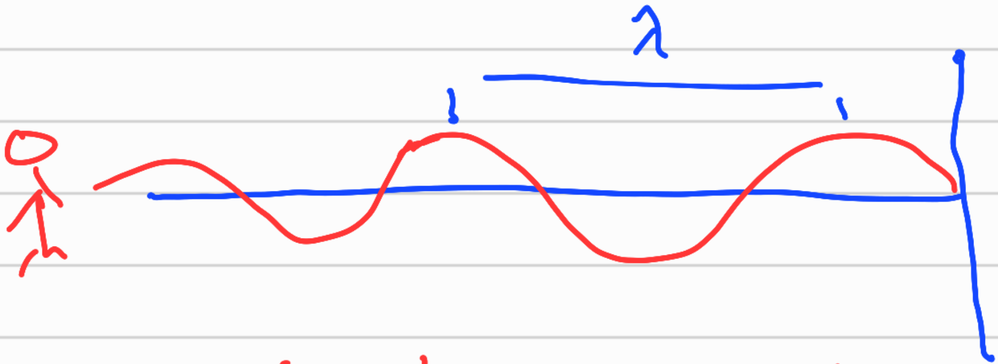
$T = \frac{1}{2} \cancel{m} \frac{p^2}{m^{\cancel{2}}}$

$i^2 = -1$

$$\langle T \rangle = -\frac{\hbar^2}{2m} \int \psi^* \frac{\partial^2 \psi}{\partial x^2} \, dx$$

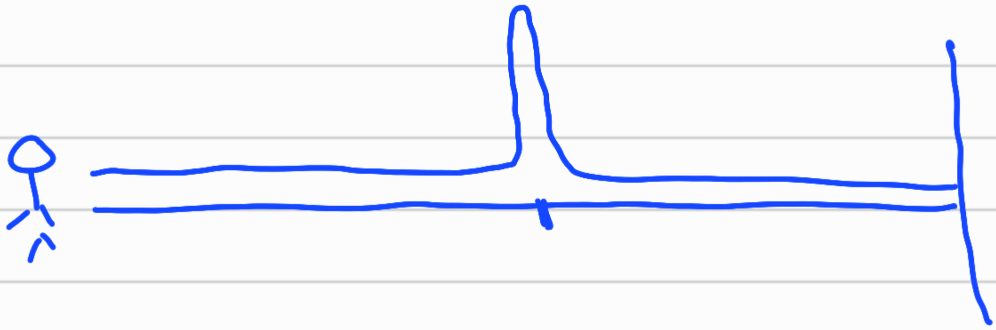
1.6 The uncertainity Principle

فرضية الشك عدم الدقة



→ we can't determine the exact Position of the wave

→ yes we can determine the wavelength



✓ We able to define more accurat position of the wave

✓ We can't accurately find the wavelength

اذا ركزنا القياس على موقع الجسم سنحصل
على قياس غير دقيق للزخم والعكس صحيح

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

x : Position $\qquad$ p : momentum

= de Broglie formula

علاقة مرتبطة بين طول الموجه وزخمها

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$$

كلما كان طول الموجة أقل كان الزخم اكبر

1.15 + 1.7 + 1.5 حل مسائل

Problem 1.5 Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t},$$ $\psi^* = Ae^{-\lambda|x|}e^{i\omega t}$

where A, λ, and ω are positive real constants. (We'll see in Chapter 2 for what potential (V) this wave function satisfies the Schrödinger equation.)

(a) Normalize Ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x\rangle+\sigma)$ and $(\langle x\rangle-\sigma)$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

a) $\int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} \psi \psi^* dx = 1$

$1 = \int_{-\infty}^{\infty} Ae^{-\lambda|x|} e^{-i\omega t} \cdot Ae^{-\lambda|x|} e^{i\omega t} dx$

$= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx$

$|x| = \begin{cases} -x & x \le 0 \\ x & x \ge 0 \end{cases}$

$= A^2 \left(\int_{-\infty}^{0} e^{2\lambda x} dx + \int_{0}^{\infty} e^{-2\lambda x} dx \right)$

$\int e^{3x}$ $\quad \frac{e^{3x}}{3}$

$1 = A^2 \left(\left. \frac{e^{2\lambda x}}{2\lambda} \right]_{-\infty}^{0} + \left. \frac{e^{-2\lambda x}}{-2\lambda} \right]_{0}^{\infty} \right)$

$1 = A^2 \left[\frac{1}{2\lambda} - 0 \right] + \left[0 - \frac{1}{-2\lambda} \right]$

$e^{-\infty} \rightsquigarrow 0$

$e^{\infty} \rightsquigarrow \infty$

$$1 = A^2 \left[\frac{1}{2\lambda} + \frac{1}{2\lambda} \right]$$

$$1 = A^2 \frac{\cancel{2}}{\cancel{2}\lambda} \qquad 1 = \frac{A^2}{\lambda}$$

$$A^2 = \lambda \qquad \boxed{A = \sqrt{\lambda}}$$

$$\psi(x,t) = \sqrt{\lambda}\, e^{-\lambda|x|} e^{-i\omega t}$$

(b) $$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = \int_{-\infty}^{\infty} x \, \psi \psi^* dx$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{i\omega t} \cdot \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{i\omega t} dx$$

$$\langle x \rangle = \lambda \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \rightarrow \underset{\int = 0}{\text{odd}}$$

$$= \text{zero}$$

$$\langle x^2 \rangle = \int x^2 |\psi|^2 dx$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{-i\omega t} \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{i\omega t} dx$$

$$\langle x^2 \rangle = \lambda \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx = 2\lambda \int_{0}^{\infty} x^2 e^{-2\lambda x} dx$$

$$\langle x^2\rangle = 2\lambda\int_0^\infty x^2 e^{-2\lambda x}$$

$u = x^2$ $\quad dv = e^{-2\lambda x}$

$du = 2x$ $\quad v = \frac{e^{-2\lambda x}}{-2\lambda}$

$$\langle x^2\rangle = 2\lambda\left(\left.\frac{x^2 e^{-2\lambda x}}{2\lambda}\right]_0^\infty + \int_0^\infty \frac{x e^{-2\lambda x}}{-\lambda}\right)$$

$uv - \int v\,du$

$[0-0]$

0

$$\langle x^2\rangle = 2\lambda\int_0^\infty \frac{x e^{-2\lambda x}}{\lambda}$$

$$\langle x^2\rangle = \frac{2\cancel{\lambda}}{\lambda}\int_0^\infty x e^{-2\lambda x}\,dx$$

$u = x$ $\quad du = 1$

$dv = e^{-2\lambda x}$ $\quad v = \frac{e^{-2\lambda x}}{-2\lambda}$

$$\langle x^2\rangle = 2\left[\left.\frac{x e^{-2\lambda x}}{2\lambda}\right]_0^\infty - \int_0^\infty \frac{e^{-2\lambda}}{-2\lambda}\right]$$

0

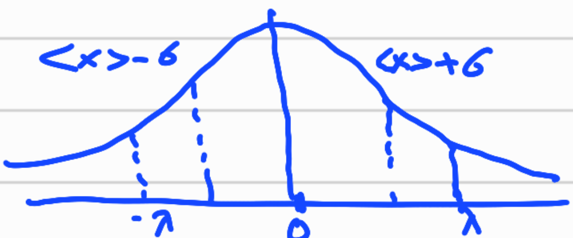
$$+\frac{\cancel{2}}{\cancel{2}\lambda}\int_0^\infty e^{-2\lambda x}\,dx = \frac{1}{\lambda}\left[\frac{e^{-2\lambda x}}{-2\lambda}\right]_0^\infty$$

$$\frac{1}{-2\lambda^2}\left[e^{-2\lambda x}\right]_0^\infty$$

$$\frac{1}{-2\lambda^2}\left[0 - e^0\right] = \frac{1}{-2\lambda^2}\left[0-1\right]$$

$$\langle x^2\rangle = +\frac{1}{2\lambda^2}$$

c) $\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{2\lambda^2} - 0} = \frac{1}{\sqrt{2}\,\lambda}$



Problem 1.7 Calculate $d\langle p\rangle/dt$. Answer:

momentum potential energ

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle. \tag{1.38}$$

This is an instance of **Ehrenfest's theorem**, which asserts that *expectation values obey the classical laws.*[19]

$$\langle p\rangle = -i\hbar \int \psi^* \left(\frac{\partial \psi}{\partial x}\right) dx$$

$$\frac{d\langle p\rangle}{dt} = -i\hbar \int \frac{d}{dt}\left[\psi^* \left(\frac{\partial \psi}{\partial x}\right)\right] dx$$

$$= -i\hbar \int \frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial x} + \psi^* \frac{\partial^2 \psi}{\partial x^2} \quad dx$$

$$= -i\hbar \left(\int \frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial x} + \cancel{\psi^* \frac{\partial \psi}{\partial t}}\Big]_{-\infty}^{\infty} - \int \frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial t}\, dx \right.$$

$$u = \psi^* \qquad dv = \frac{\partial^2 \psi}{\partial x^2}$$

$$du = \frac{d\psi^*}{dx}dx \qquad v = \frac{\partial \psi}{\partial t}$$

$$= -i\hbar \int \frac{\partial \psi^*}{\partial t}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial t}\frac{\partial \psi^*}{\partial x}\, dx$$

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right]$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{1}{i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*\right]$$

$$= -\cancel{i\hbar} \int \frac{-1}{\cancel{i\hbar}}\left[\frac{-\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*\right]\frac{\partial \psi}{\partial x} - \frac{1}{\cancel{i\hbar}}\left[\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right]\frac{\partial \psi^*}{\partial x}\, dx$$

$$\int \frac{-\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi^*}{\partial x} + V\psi^*\frac{\partial\psi}{\partial x} + V\psi\frac{\partial\psi^*}{\partial x}$$

$$\int \frac{-\hbar^2}{2m}\cancel{\left[\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi^*}{\partial x}\right]} + V\psi^*\frac{\partial\psi}{\partial t} + V\psi\frac{\partial\psi^*}{\partial x}$$

zero

$$\int V\psi^*\frac{\partial\psi}{\partial t} + V\psi\frac{\partial\psi^*}{\partial x}\,dx$$

$$u = V\psi^* \qquad dv = \frac{\partial\psi}{\partial t}$$

$$du = \frac{\partial V}{\partial x}\psi^* + V\frac{\partial\psi}{\partial x} \qquad v = \psi$$

$$\cancel{V\psi\psi^*} - \int \psi\left(\frac{\partial V}{\partial x}\psi^* + V\frac{\partial\psi}{\partial x}\right) + \int V\psi\frac{\partial\psi^*}{\partial x}$$

$$= \int -\psi\frac{\partial V}{\partial x}\psi^* - \cancel{\int V\psi\frac{\partial\psi}{\partial x}} + \cancel{\int V\psi\frac{\partial\psi}{\partial x}}$$

$$= \int -\psi\frac{\partial V}{\partial x}\psi^*$$

$$= \int \psi^*\frac{-\partial V}{\partial x}\psi\,dx = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

Problem 1.15 Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \Psi_2 \, dx = 0$$

for any two (normalizable) solutions to the Schrödinger equation (with the same $V(x)$), Ψ_1 and Ψ_2.

$$\int_{-\infty}^{\infty} \frac{d}{dt} \psi_1^* \psi_2 \, dx = \int \frac{d\psi_1^*}{dt}\psi_2 + \frac{d\psi_2}{dt}\psi_1^* \, dx$$

$$\frac{d\psi}{dt} = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right] =$$

$$\frac{d\psi}{dt} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V\psi \quad ✓$$

$$\frac{d\psi^*}{dt} = \frac{-i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*$$

$$\int\left[\frac{-i\hbar}{2m}\frac{\partial^2\psi_1^*}{\partial x^2} + \cancel{\frac{i}{\hbar}V\psi_1^*}\right]\psi_2 + \left[\frac{i\hbar}{2m}\frac{\partial^2\psi_2}{\partial y^2} \cancel{- \frac{i}{\hbar}V\psi_2}\right]\psi_1^*$$

$$\int_{-\infty}^{\infty}\frac{i\hbar}{2m}\left[-\frac{\partial^2\psi_1^*}{\partial x}\psi_2 + \frac{\partial^2\psi_2}{\partial x}\psi_1^*\right] \rightarrow$$

تكامل بالاجزاء

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$$= 0$$

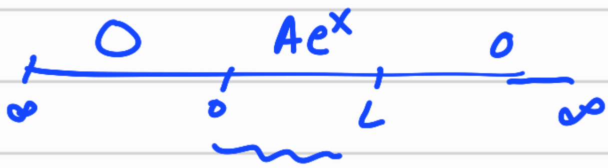
سؤال اضافي

suppose that it is determined that a particle has a wave function given by

$$\psi = \begin{cases} Ae^{x} & 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

a) Normalize to evaluate A

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$



$$1 = \int_0^L A^2 e^{2x} dx = A^2 \int_0^L e^{2x} dx$$

$$1 = A^2 \frac{e^{2x}}{2}\Big]_0^L = \frac{A^2}{2} e^{2x}\Big]_0^L$$

$$1 = \frac{A^2}{2}\left(e^{2L} - 1\right) \qquad \frac{A^2(e^{2L}-1)}{2} = 1$$

$$A^2 = \frac{2}{e^{2L}-1} \qquad A = \sqrt{\frac{2}{2^L - 1}}$$

سؤال اضافي

(2)

a) Normalize the function $e^{i(kx-\omega t)}$ in the region for 0 to 3a

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = \int_0^{3a} \psi \psi^* dx$$

$$1 = \int A e^{i(kx-\omega t)} A e^{-i(kx-\omega t)} dx$$

$$1 = A^2 \int_0^{3a} 1 \, dx$$

$$1 = A^2 \left[x \right]_0^{3a} \qquad 1 = A^2 (3a)$$

$$A^2 = \frac{1}{3a} \qquad A = \sqrt{\frac{1}{3a}}$$

b) find $\langle x \rangle$ $\langle x^2 \rangle$ σ

$$\langle x \rangle = \int x |\psi|^2 dx$$

$$\langle x \rangle = \int x \frac{1}{3a} e^{-i(\cancel{kx}-\omega t)} e^{i(\cancel{kx}-\omega t)} dx$$

$$\langle x \rangle = \frac{1}{3a} \int_0^{3a} x = \frac{1}{3a} \left. \frac{x^2}{2} \right]_0^{3a}$$

$$\langle x \rangle = \frac{1}{3a}\left[\frac{9a^2 - 0}{2}\right] = \frac{3a}{2}$$

$$\langle x^2 \rangle = \frac{1}{3a}\int_0^{3a} x^2 = \frac{1}{3a}\left.\frac{x^3}{3}\right]_0^{3a}$$

$$\langle x^2 \rangle = \frac{1}{3a}\left[\frac{(3a)^3}{3} - 0\right]$$

$$\langle x^2 \rangle = 3a^2$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma = \sqrt{3a^2 - \frac{9}{4}a^2}$$

$$\sigma = \sqrt{\frac{12}{4}a^2 - \frac{9}{4}a^2} = \sqrt{\frac{3a^2}{4}}$$

Problem 1.9 A particle of mass m has the wave function

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]},$$

where A and a are positive real constants.

(a) Find A.

(b) For what potential energy function, $V(x)$, is this a solution to the Schrödinger equation?

(c) Calculate the expectation values of x, x^2, p, and p^2.

(d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

$$|\psi|^2 = \psi\psi^* = Ae^{-a[mx^2/\hbar + \cancel{it}]} \cdot Ae^{-a[mx^2/\hbar - \cancel{it}]}$$

$$|\psi|^2 = A^2 e^{-2amx^2/\hbar}$$

$$\int_0^\infty e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

1) find A

$$1 = \int_{-\infty}^{\infty} |\psi|^2 = A^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\, dx$$

$$= 2A^2 \int_0^\infty e^{-x^2\left(\frac{2am}{\hbar}\right)} = \cancel{2}A^2 \frac{1}{\cancel{2}}\sqrt{\frac{\pi\hbar}{2am}}$$

$$1 = A^2 \sqrt{\frac{\pi\hbar}{2am}}$$

$$A^2 = \sqrt{\frac{2am}{\pi\hbar}} \qquad A = \sqrt[4]{\frac{2am}{\pi\hbar}}$$

$$\psi = \sqrt[4]{\frac{2am}{\pi\hbar}}\, e^{-a[mx^2/\hbar + it]}$$

✓

b) تطبيق معادلة sh وضعية V

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi$$

$$V = \frac{\hbar}{i\psi}\left[\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial t}\right]$$

$$V = \frac{\hbar\, e^{a[mx^2/\hbar + it]}}{i\sqrt[4]{\frac{2am}{\pi\hbar}}}\left[\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\left[\sqrt[4]{\frac{2am}{\hbar\pi}}\, e^{-a[\frac{mx^2}{\hbar}+it]}\right] - \frac{\partial}{\partial t}\sqrt[4]{\frac{2am}{\hbar\pi}}\, e^{-a[\frac{mx^2}{\hbar}+it]}\right]$$

$$V = \frac{\hbar}{i}\sqrt[4]{\frac{\pi\hbar}{2am}}\, e^{a[\frac{mx^2}{\hbar}+it}\left[\frac{i\hbar}{2m}\sqrt[4]{\frac{2am}{\hbar\pi}}\frac{\partial}{\partial x} e^{-a[\frac{mx^2}{\hbar}+it]}\left[\frac{-a2mx}{\hbar}\right] - \sqrt[4]{\frac{2am}{\hbar\pi}}\, e^{-a[\frac{mx^2}{\hbar}+it]}(-ai)\right.$$

$$V = \frac{\hbar}{i} e^{a[\frac{mx^2}{\hbar}+it]}\left[\frac{i\hbar}{2m} - \left[\frac{-2^2 m^2 a^2 x^2}{\hbar^2}\right] e^{-a[\frac{mx^2}{\hbar}-it]} + \left[\frac{-a2m}{\hbar}\right] e^{-a[\frac{mx^2}{\hbar}-it]}\right.$$

$$+ai\, e^{-a[\frac{mx^2}{\hbar}+it]}$$

$$V = \hbar\, e^{\cancel{a[\frac{mx^2}{\hbar}+it]}}\left[\frac{\cancel{\hbar}}{2\cancel{m}}\frac{2ma}{\hbar}\left[-1 + \frac{2ma}{\hbar}x^2\right] e^{-a[\cancel{\frac{mx^2}{\hbar}}-it}\right]$$

$$+a e^{-a[\cancel{\frac{mx^2}{\hbar}} + it]}$$

$$V = \hbar\left(\left[a\left(-1 + \frac{2max^2}{\hbar}\right)\right] + a\right)$$

$$V = \hbar\left(-\cancel{a} + \frac{2ma^2x^2}{\hbar} + \cancel{a}\right) = 2ma^2x^2$$

c) $\langle x\rangle = \int x|\psi|^2$

$$\langle x\rangle = \int \cancel{x}\, A^2 e^{-2amx^2/\hbar}\, dx$$

$$\langle x\rangle = \sqrt{\frac{2am}{\pi\hbar}} \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar}\, dx \quad \checkmark$$

$$\langle x\rangle = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2|\psi|^2 dy$$

$$\int_0^{\infty} x^2 e^{-ax^2} = \frac{1}{4a}\left(\frac{\pi}{a}\right)^{1/2}.$$

$$\langle x^2\rangle = \sqrt{\frac{2am}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-2amx^2/\hbar}\, dx$$

$$\langle x^2\rangle = 2\sqrt{\frac{2am}{\pi\hbar}} \int_0^{\infty} x^2 e^{-x^2\left(\frac{2am}{\hbar}\right)}$$

$$\langle x^2\rangle = \cancel{2}\sqrt{\cancel{\frac{2am}{\pi\hbar}}}\left[\frac{1}{4\left(\frac{\cancel{2}am}{\hbar}\right)}\frac{\sqrt{\cancel{\pi}}}{\cancel{\sqrt{\frac{2am}{\hbar}}}}\right] = \frac{\hbar}{4ma}$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^*\left(i\hbar\frac{\partial}{\partial x}\right)\psi$$

$$\langle p \rangle = i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x}\, dx$$

$$\langle p \rangle = i\hbar \int_{-\infty}^{\infty} \sqrt[4]{\frac{2am}{\hbar}}\, e^{-a\left[\frac{mx^2}{\hbar} - it\right]} \frac{\partial}{\partial x} \sqrt[4]{\frac{2am}{\hbar}}\, e^{-a\left[\frac{mx^2}{\hbar} + it\right]}\, dx$$

$$\langle p \rangle = i\hbar \sqrt{\frac{2am}{\hbar}} \int e^{-a\left[\frac{mx^2}{\hbar} - it\right]} \cdot e^{-a\left[\frac{mx^2}{\hbar} + it\right]} \left(\frac{-a2mx}{\hbar}\right)$$

$$\langle p \rangle = i\hbar \sqrt{\frac{2am}{\hbar}} \left(-\frac{2am}{\hbar}\right) \underbrace{\int_{-\infty}^{\infty} x e^{-2a\frac{mx^2}{\hbar}}}_{0}$$

$$\langle p \rangle = 0$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \psi\, dx$$

$$= \hbar a m$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{4ma} - 0} = \sqrt{\frac{\hbar}{4ma}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar a m - 0} = \sqrt{\hbar a m}$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{4\cancel{ma}}}\, \sqrt{\hbar \cancel{am}} = \frac{\sqrt{\hbar}\sqrt{\hbar}}{\sqrt{4}} = \frac{\hbar}{2}$$

Problem 1.9 A particle of mass m has the wave function

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]},$$

where A and a are positive real constants.

(a) Find A.

(b) For what potential energy function, $V(x)$, is this a solution to the Schrödinger equation?

(c) Calculate the expectation values of x, x^2, p, and p^2.

(d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

$$\langle p^2 \rangle = \int \psi^* \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \psi$$

$$= -\hbar^2 \int \psi^* \frac{\partial^2}{\partial x^2} \psi$$

$$= -\hbar^2 \int \sqrt[4]{\frac{2am}{\pi\hbar}}\, e^{-a\left[\frac{mx^2}{\hbar} - it\right]} \frac{\partial^2}{\partial x^2} \sqrt[4]{\frac{2am}{\pi\hbar}}\, e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$$

$$= -\hbar^2 \sqrt[2]{\frac{2am}{\pi\hbar}} \int e^{-a\left[\frac{mx^2}{\hbar} - it\right]} \underbrace{\frac{\partial^2}{\partial x^2} e^{-a\left[\frac{mx^2}{\hbar} + it\right]}}$$

$$= -\hbar^2 \sqrt[2]{\frac{2am}{\pi\hbar}} \int e^{-a\left[\frac{mx^2}{\hbar} - it\right]}$$

$$\Rightarrow \frac{-2amx}{\hbar} \underbrace{e^{-a\left[\frac{mx^2}{\hbar} + it\right]}}$$

$$+ \frac{4a^2m^2x^2}{\hbar^2} e^{-a\left[\frac{mx^2}{\hbar} + it\right]} + \frac{-2am}{\hbar} e^{-a\left[\frac{mx^2}{\hbar} + it\right]}$$

$$\underbrace{\left\{ \frac{-2am}{\hbar} e^{-a\left[\frac{mx^2}{\hbar}+it\right]} \left[\frac{-2amx^2}{\hbar} + 1 \right] \right\}}$$

$$= -\frac{\hbar^2}{2}\sqrt{\frac{2am}{\pi\hbar}} \int e^{-a\left[\frac{mx^2}{\hbar}-it\right]} \left[\frac{-2am}{\hbar} e^{-a\left[\frac{mx^2}{\hbar}+it\right]} \left(1 - \frac{2amx^2}{\hbar}\right) \right]$$

$$= -\hbar^2 \sqrt{\frac{2am}{\pi\hbar}} \left(\frac{-2am}{\hbar}\right) \int e^{\frac{-2amx^2}{\hbar}} \left(1 - \frac{2amx^2}{\hbar}\right)$$

$$= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}} \left(\frac{2am}{\hbar}\right) \left[2\int_0^\infty e^{\frac{-2amx^2}{\hbar}} - \left[\int \frac{2amx^2}{\hbar} e^{\frac{-2amx^2}{\hbar}} \right] \right]$$

$$= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}} \left(\frac{2am}{\hbar}\right) \left[2\int_0^\infty e^{\frac{-2amx^2}{\hbar}} - \frac{4am}{\hbar} \int_0^\infty x^2 e^{-\frac{2amx^2}{\hbar}} \right]$$

Table of Useful Integrals, etc.

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2} \qquad \int_0^\infty x e^{-ax^2}\,dx = \frac{1}{2a}$$

$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{4a}\left(\frac{\pi}{a}\right)^{1/2} \qquad \int_0^\infty x^3 e^{-ax^2}\,dx = \frac{1}{2a^2}$$

$$= \hbar^2 \sqrt{\frac{2am}{\pi\hbar}} \left(\frac{2am}{\hbar}\right) \cdot \left(\cancel{2}\, \frac{1}{\cancel{2}} \sqrt{\frac{\pi\hbar}{2am}} - \frac{\cancel{4am}}{\cancel{\hbar}} \frac{\cancel{\hbar}}{4(2\cancel{am})} \sqrt{\frac{\pi\hbar}{2am}} \right)$$

$$= \hbar^{\cancel{2}} \cancel{\sqrt{\frac{2am}{\pi\hbar}}}\ \frac{\cancel{2}am}{\cancel{\hbar}} \left[\cancel{\frac{1}{2}} \cancel{\sqrt{\frac{\pi\hbar}{2am}}} \right]$$

$$\boxed{am\hbar}$$

Problem 1.16 A particle is represented (at time $t = 0$) by the wave function

$$\Psi(x,0) = \begin{cases} A\left(a^2 - x^2\right), & -a \le x \le +a, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the normalization constant A.
(b) What is the expectation value of x?
(c) What is the expectation value of p? (Note that you *cannot* get it from $\langle p \rangle = m d\langle x \rangle / dt$. Why not?)

a) $$1 = \int_{-\infty}^{\infty} |\psi^2|\, dx = \int_{-a}^{a} A^2 (a^2 - x^2)^2 dx$$

$$1 = 2A^2 \int_0^a a^4 - 2a^2x^2 + x^4 \quad dx$$

$$1 = 2A^2 \left[a^4 x - \frac{2}{3} a^2 x^3 + \frac{x^5}{5} \right]_0^a = 2A^2 \left[a^5 - \frac{2}{3} a^5 + \frac{a^5}{5} \right]$$

$$1 = 2A^2 \left[\frac{15a^5}{15} - \frac{10a^5}{15} + \frac{3a^5}{15} \right] \Rightarrow 1 = 2A^2 \; \frac{8a^5}{15}$$

$$1 = \frac{16}{15} A^2 a^5$$

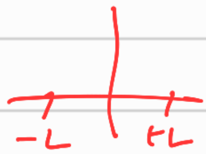
$$A^2 = \frac{15}{16a^5} \qquad A = \sqrt{\frac{15}{16a^5}}$$

b) $$\langle x \rangle = \int x |\psi^2| dx = A^2 \int_{-a}^{a} x (a^2 - x^2)^2 dx$$

$$= \frac{15}{16} a^5 \int_{-a}^{a} (x)(a^4 + 2a^2x^2 + x^4)\; dx$$

odd

$$= 0$$



c) $\langle p \rangle = \int^{\infty} \psi^* \, -i\hbar \frac{\partial}{\partial x} \, \psi$

$$\langle p \rangle = -i\hbar A^2 \int_{-a}^{a} (a^2 - x^2) \frac{\partial}{\partial x} (a^2 - x^2) \, dx$$

$$\langle p \rangle = -i\hbar A^2 \int (a^2 - x^2)(-2x) \, d$$

$$\langle p \rangle = +2\hbar A^2 \int_{-a}^{a} x(a^2 - x^2)$$ → even

= zero

(d) Find the expectation value of x^2.
(e) Find the expectation value of p^2.
(f) Find the uncertainty in x (σ_x). $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$
(g) Find the uncertainty in p (σ_p). $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$
(h) Check that your results are consistent with the uncertainty principle.

c) $\langle x^2 \rangle = \int x^2 |\psi^2| \, dx$

$$\frac{15}{16a^5} \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx$$

$$\frac{15}{16a^5} \int_{-a}^{a} (x^2 a^4 - 2a^2 x^4 + x^6) \, dx$$

$$2 \, \frac{15}{16a^5} \int_{0}^{a} x^2 a^4 - 2a^2 x^4 + x^6$$

$$\frac{15}{8a^5}\left[\frac{x^3}{3}a^4 - 2a^2\frac{x^5}{5} + \frac{x^7}{7}\right]_0^a$$

$$\frac{15}{8a^5}\left[\frac{a^7}{3} - \frac{2a^7}{5} + \frac{a^7}{7}\right]$$

$$\frac{15}{8}a^7\left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right]$$

$$\frac{a^7}{8}\left[5 - 6 + \frac{15}{7}\right]$$

$$\frac{a^7}{8}\left[\frac{-7}{7} + \frac{15}{7}\right] = \frac{a^7\cancel{8}}{\cancel{8}\times 7} = \frac{a^7}{7}$$

e) $$\langle p^2\rangle = \int_{-a}^{a}\psi(-i\hbar)^2\frac{\partial^2}{\partial x^2}\psi \qquad (i)^2=-1$$

$$\langle p^2\rangle = \hbar^2\int_{-a}^{a}A(a^2-x^2)\,\frac{\partial^2}{\partial x^2}A\underbrace{(a^2-x^2)}_{0-2x}$$

$$\langle p^2\rangle = \hbar^2A^2\int_{-a}^{a}(a^2-x^2)(-2)$$

$$\langle p^2\rangle = (-2)\,2\hbar^2\,\frac{15}{16a^5}\int_{0}^{a}(a^2-x^2)\,dx$$

$$= -\hbar^2\frac{15}{4a^5}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= -\frac{15\hbar^2}{4a^5}\left[a^3 - \frac{a^3}{3}\right]$$

$$\langle p^2\rangle = -\frac{\overset{5}{\cancel{15}}\hbar^2}{\underset{2}{\cancel{4}}a^5}\,\frac{\cancel{2}a^3}{\cancel{3}} = -\frac{5\hbar^2}{2a^2}$$

f) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$$\sigma_x = \sqrt{\frac{a^2}{7}}$$

g) $\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$

$$= \sqrt{\frac{5\hbar^2}{2a^2}}$$

h) $\sigma_x \sigma_x \geq \frac{\hbar}{2}$

$$\sqrt{\frac{a^2}{7}}\sqrt{\frac{5\hbar^2}{2a^2}} \overset{?}{\geq} \frac{\hbar}{2} \longrightarrow 0.5\hbar$$

$$\sqrt{\frac{\cancel{a^2}\,5\,\hbar^2}{14\,\cancel{a^2}}} = \sqrt{\frac{5}{14}}\,\hbar = 0.59\hbar$$

ملخص قوانين شابتر ١

$$\Rightarrow \int \psi^* \psi \, dx = \int |\psi|^2 dx = 1$$

$$\Rightarrow \langle x \rangle = \int x |\psi^2| \, dx$$

$$\Rightarrow \langle x^2 \rangle = \int x^2 |\psi^2| \, dx$$

$$\Rightarrow \langle p \rangle = \int \psi^* i\hbar \frac{\partial}{\partial x} \psi \, dx$$

$$\Rightarrow \langle p^2 \rangle = \int \psi \left(i\hbar \frac{\partial}{\partial x}\right)^2 \psi \, dx$$

$$\Rightarrow \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Rightarrow \sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$\Rightarrow \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

جدول تكاملات وقوانين مهمة

Table of Useful Integrals, etc.

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\left(\frac{\pi}{a}\right)^{1/2} \qquad \int_0^\infty x e^{-ax^2}\,dx = \frac{1}{2a}$$

$$\int_0^\infty x^2 e^{-ax^2}\,dx = \frac{1}{4a}\left(\frac{\pi}{a}\right)^{1/2} \qquad \int_0^\infty x^3 e^{-ax^2}\,dx = \frac{1}{2a^2}$$

$$\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\left(\frac{\pi}{a}\right)^{1/2} \qquad \int_0^\infty x^{2n+1} e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}$$

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

Integration by Parts:

$$\int_a^b U\,dV = \left[UV\right]_a^b - \int_a^b V\,dU$$ U and V are functions of x. Integrate from x = a to x = b

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$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \sin^3(ax)\,dx = -\frac{1}{a}\cos(ax) + \frac{1}{3a}\cos^3(ax)$$

$$\int \sin^4(ax)\,dx = \frac{3x}{8} - \frac{3\sin(2ax)}{16a} - \frac{\sin^3(ax)\cos(ax)}{4a}$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin\left[(a-b)x\right]}{2(a-b)} - \frac{\sin\left[(a+b)x\right]}{2(a+b)} \quad \text{where } a^2 \neq b^2$$

$$\int \cos(ax)\cos(bx)\,dx = \frac{\sin\left[(a-b)x\right]}{2(a-b)} + \frac{\sin\left[(a+b)x\right]}{2(a+b)}$$

$$\int \sin(ax)\cos(bx)\,dx = \frac{-\cos\left[(a-b)x\right]}{2(a-b)} - \frac{\cos\left[(a+b)x\right]}{2(a+b)}$$

$$\int x\sin^2(ax)\,dx = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin(2ax) - \frac{x\cos(2ax)}{4a^2}$$

$$\int x\sin(ax)\sin(bx)\,dx = \frac{\cos\left[(a-b)x\right]}{2(a-b)^2} - \frac{\cos\left[(a+b)x\right]}{2(a+b)^2} + \frac{x\sin\left[(a-b)x\right]}{2(a-b)^2} - \frac{x\sin\left[(a+b)x\right]}{2(a+b)^2}$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$$

$$\int x\cos(ax)\,dx = \frac{x\sin(ax)}{a} + \frac{\cos(ax)}{a^2}$$

$$\int \cos(ax)\,dx = \frac{\sin(ax)}{a}$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x^2\cos^2(ax)\,dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin(2ax) + \frac{x\cos(2ax)}{4a^2}$$

$$\int \cos(bx)e^{-ax^2}\,dx = \frac{e^{ax}}{(a^2+b^2)}\left[a\cos(bx) + b\sin(bx)\right]$$

Taylor Series:

$$\sum_{n=0}^{n=\infty} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n$$

Geometric Series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Euler's Formula:

$e^{i\phi} = \cos\phi + i\sin\phi$

Quadratic Equation and other higher order polynomials:

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^4 + bx^2 + c = 0$

$$x = \pm\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$$

General Solution for a Second Order Homogeneous Differential Equation with Constant Coefficients:

If: $y'' + py' + qy = 0$

Assume a solution for y:

$y = e^{sx} \quad y' = se^{sx} \quad y'' = s^2e^{sx}$

$\therefore \quad s^2e^{sx} + pse^{sx} + qe^{sx} = 0$

$and \quad s^2 + ps + q = 0$

$Hence \quad y = c_1e^{s_1x} + c_2e^{s_2x}$

Conversions from spherical polar coordinates into Cartesian coordinates:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$x = r\cos\theta$

$dv = r^2\sin\theta dr d\theta d\phi$

$0 < r < \infty$

$0 < \theta < \pi$

$0 < \phi < 2\pi$

Commutator Identities:

$$\left[\hat{A},\hat{B}\right] = -\left[\hat{B},\hat{A}\right]$$

$$\left[\hat{A},\hat{A}^n\right] = 0 \quad n = 1,2,3,\cdots$$

$$\left[k\hat{A},\hat{B}\right] = \left[\hat{A},k\hat{B}\right] = k\left[\hat{A},\hat{B}\right]$$

$$\left[\hat{A}+\hat{B},\hat{C}\right] = \left[\hat{A},\hat{C}\right] + \left[\hat{B},\hat{C}\right]$$

$$\left[\hat{A},\hat{B}\hat{C}\right] = \left[\hat{A},\hat{B}\right]\hat{C} + \hat{B}\left[\hat{A},\hat{C}\right]$$

$$\left[\hat{A}\hat{B},\hat{C}\right] = \left[\hat{A},\hat{C}\right]\hat{B} + \hat{A}\left[\hat{B},\hat{C}\right]$$

Creation and Annihilation Operators

$$l_\pm\left|l,m_l,s,m_s\right\rangle = \left(l\left(l+1\right) - m_l\left(m_l \pm 1\right)\right)^{1/2}\hbar\left|l,m_{l\pm1},s,m_s\right\rangle$$

$$s_\pm\left|l,m_l,s,m_s\right\rangle = \left(s\left(s+1\right) - m_s\left(m_s \pm 1\right)\right)^{1/2}\hbar\left|l,m_l,s,m_{s\pm1}\right\rangle$$

$$j_\pm\left|j,m_j\right\rangle = \left(j\left(j+1\right) - m_j\left(m_j \pm 1\right)\right)^{1/2}\hbar\left|j,m_{j\pm1}\right\rangle$$

Atomic Units:

Quantity	Atomic unit in cgs or other units	Values of some atomic properties in atomic units (a.u.)
Mass	$m_e = 9.109534 \times 10^{-28}$ g	Mass of electron = 1 a.u.
Length	$a_0 = 4\pi\varepsilon_0\hbar^2/m_e e^2$ $= 0.52917706 \times 10^{-10}$ m (= 1 bohr)	Most probable distance of 1s electron from nucleus of H atom = 1 a.u.
Time	$\tau_0 = a_0\hbar/e^2$ $= 2.4189 \times 10^{-17}$ sec	Time for 1s electron in H atom to travel one bohr = 1 a.u.
Charge	$e = 4.803242 \times 10^{-10}$ esu $= 1.6021892 \times 10^{-19}$ coulomb	Charge of electron = -1 a.u.
Energy	$e^2/4\pi\varepsilon_0 a_0 = 4.359814 \times 10^{-18}$ J (= 27.21161 eV ≡ 1 hartree)	Total energy of 1s electron in H atom = $-1/2$ a.u.
Angular momentum	$\hbar = h/2\pi$ $= 1.0545887 \times 10^{-34}$ J sec	Angular momentum for particle in ring = 0, 1, 2, ... a.u.
Electric field strength	$e/a_0{}^2 = 5.1423 \times 10^9$ V/cm	Electric field strength at distance of 1 bohr from proton = 1 a.u.

Operators:

TABLE 4.1
Classical-mechanical observables and their corresponding quantum-mechanical operators.

Observable		Operator	
Name	Symbol	Symbol	Operation
Position	x	$\hat{X}$	Multiply by x
	$\mathbf{r}$	$\hat{\mathbf{R}}$	Multiply by $\mathbf{r}$
Momentum	p_x	$\hat{P}_x$	$-i\hbar\dfrac{\partial}{\partial x}$
	$\mathbf{p}$	$\hat{\mathbf{P}}$	$-i\hbar\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)$
Kinetic energy	K_x	$\hat{K}_x$	$-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}$
	K	$\hat{K}$	$-\dfrac{\hbar^2}{2m}\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)$ $=-\dfrac{\hbar^2}{2m}\nabla^2$
Potential energy	$V(x)$	$\hat{V}(\hat{x})$	Multiply by $V(x)$
	$V(x, y, z)$	$\hat{V}(\hat{x}, \hat{y}, \hat{z})$	Multiply by $V(x, y, z)$
Total energy	E	$\hat{H}$	$-\dfrac{\hbar^2}{2m}\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)$ $+V(x, y, z)$ $=-\dfrac{\hbar^2}{2m}\nabla^2+V(x, y, z)$
Angular momentum	$L_x = yp_z - zp_y$	$\hat{L}_x$	$-i\hbar\left(y\dfrac{\partial}{\partial z}-z\dfrac{\partial}{\partial y}\right)$
	$L_y = zp_x - xp_z$	$\hat{L}_y$	$-i\hbar\left(z\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial z}\right)$
	$L_z = xp_y - yp_x$	$\hat{L}_z$	$-i\hbar\left(x\dfrac{\partial}{\partial y}-y\dfrac{\partial}{\partial x}\right)$