



Electromagnetic Field: Phy321

Exercise Ch1: Vector Analysis

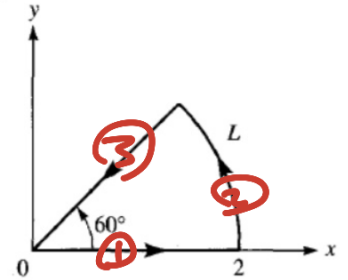
PRACTICE EXERCISE 3.2

Calculate the circulation of

$$\mathbf{A} = \rho \cos \phi \mathbf{a}_\rho + z \sin \phi \mathbf{a}_\phi$$

around the edge L of the wedge defined by $0 \leq \rho \leq 2$, $0 \leq \phi \leq 60^\circ$, $z = 0$ and shown in Figure 3.11.

Answer: 1.



$$d\mathbf{l} = d\rho \hat{\mathbf{a}}_\rho + \rho d\phi \hat{\mathbf{a}}_\phi + dz \hat{\mathbf{a}}_z$$

$$\int_L \mathbf{A} \cdot d\mathbf{l} = \int^{\textcircled{1}} + \int^{\textcircled{2}} + \int^{\textcircled{3}} = 2 + 0 - 1 = \boxed{1}$$

$$\begin{aligned} \textcircled{1} \int &= \int \mathbf{A} \cdot d\mathbf{l} = \int_0^2 \rho \cos \phi \hat{\mathbf{a}}_\rho \cdot d\rho \hat{\mathbf{a}}_\rho \\ &= \int_0^2 \rho \cos \phi d\rho = \cos 0 \int_0^2 \rho d\rho = \left[\frac{\rho^2}{2} \right]_0^2 \end{aligned}$$

$$\textcircled{1} \int = 2$$

$$\textcircled{2} \int = \int \mathbf{A} \cdot d\mathbf{l} = \int z \sin \phi \mathbf{a}_\phi \cdot \rho d\phi \mathbf{a}_\phi$$

$$\textcircled{2} \int = 0$$

$$\textcircled{3} \int_2^0 \rho \cos \phi d\rho = \cos 60^\circ \int_2^0 \rho d\rho = \frac{1}{2} \left[\frac{\rho^2}{2} \right]_2^0 = \boxed{-1}$$

PRACTICE EXERCISE 3.3

Determine the gradient of the following scalar fields:

(a) $U = x^2y + xyz$

(b) $V = \rho z \sin \phi + z^2 \cos^2 \phi + \rho^2$

$$a) \nabla U = \frac{\partial U}{\partial x} \hat{a}_x + \frac{\partial U}{\partial y} \hat{a}_y + \frac{\partial U}{\partial z} \hat{a}_z$$

$$\nabla U = (2xy + yz) \hat{a}_x + (x^2 + xz) \hat{a}_y + (xy) \hat{a}_z$$

$$b) \nabla V = \frac{\partial V}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi + \frac{\partial V}{\partial z} \hat{a}_z$$

$$= [z \sin \phi + 2\rho] \hat{a}_\rho + \frac{1}{\rho} [\rho z^2 \cos \phi - z^2 \underline{2 \cos \phi \sin \phi}] \hat{a}_\phi \\ + [\rho \sin \phi + 2z \cos^2 \phi] \hat{a}_z$$

$$= [z \sin \phi + 2\rho] \hat{a}_\rho + [z^2 \cos \phi - \frac{z^2}{\rho} \sin 2\phi] \hat{a}_\phi \\ + [\rho \sin \phi + 2z \cos \phi] \hat{a}_z$$

PRACTICE EXERCISE 3.4

Given $\Phi = xy + yz + xz$, find gradient Φ at point $(1, 2, 3)$ and the directional derivative of Φ at the same point in the direction toward point $(3, 4, 4)$.

Answer: $5\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z, 7$.

①

$$\nabla \Phi = (y+z)\mathbf{a}_x + (x+z)\mathbf{a}_y + (y+x)\mathbf{a}_z$$

at $(1, 2, 3)$

$$\nabla \Phi = 5\hat{\mathbf{a}}_x + 4\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z$$

$(5, 4, 3)$

② Directional Derivative

A $(1, 2, 3)$

B $(3, 4, 4)$

Distance between A and B

$$\overline{AB} = (3-1, 4-2, 4-3)$$
$$= 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$$

$$\hat{\mathbf{a}} = \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{3}\mathbf{a}_x + \frac{2}{3}\mathbf{a}_y + \frac{1}{3}\mathbf{a}_z$$

$(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

$$\frac{d\Phi}{dt} = \nabla \Phi \cdot \hat{\mathbf{a}} =$$

$$\frac{10}{3} + \frac{8}{3} + \frac{3}{3} = \frac{21}{3} = \boxed{7}$$

PRACTICE EXERCISE 3.6

Determine the divergence of the following vector fields and evaluate them at the specified points.

(a) $\mathbf{A} = yz\mathbf{a}_x + 4xy\mathbf{a}_y + y\mathbf{a}_z$ at $(1, -2, 3)$

(b) $\mathbf{B} = \rho z \sin \phi \mathbf{a}_\rho + 3\rho z^2 \cos \phi \mathbf{a}_\phi$ at $(5, \pi/2, 1)$

$$a) \quad \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= 0 + 4x + 0 = 4x$$

$$\text{at } (1, -2, 3) \quad \nabla \cdot \mathbf{A} = 4(1) = 4$$

$$b) \quad \nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 z \sin \phi) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (3\rho z^2 \cos \phi)$$

$$= 2z \sin \phi - 3z^2 \sin \phi$$

$$\nabla \cdot \mathbf{B} = (2 - 3z) z \sin \phi$$

$$\text{at } (5, \pi/2, 1) \quad \nabla \cdot \mathbf{B} = (2 - 3(1)) \sin \pi/2$$

$$\nabla \cdot \mathbf{B} = -1$$

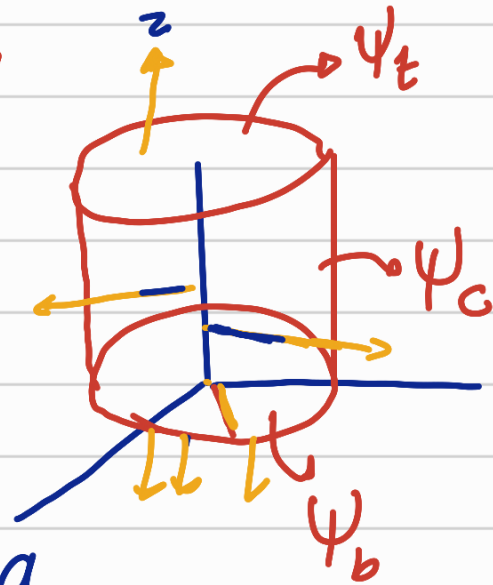
PRACTICE EXERCISE 3.7

Determine the flux of $\mathbf{D} = \rho^2 \cos^2 \phi \mathbf{a}_\rho + z \sin \phi \mathbf{a}_\phi$ over the closed surface of the cylinder $0 \leq z \leq 1, \rho = 4$. Verify the divergence theorem for this case.

Answer: 64π .

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{s} = \int_V \nabla \cdot \mathbf{D} dV$$

$$\Psi = \oint \mathbf{D} \cdot d\mathbf{s} = \Psi_t + \Psi_b + \Psi_c$$



$$\Psi_t \quad z=1 \quad d\mathbf{s} = \rho d\rho d\phi \mathbf{a}_z$$

$$\Psi_t = \int (\rho^2 \cos^2 \phi \mathbf{a}_\rho + z \sin \phi \mathbf{a}_\phi) \cdot (\rho d\rho d\phi \mathbf{a}_z)$$

$$\Psi_t = 0$$

$$\Psi_b \quad z=1$$

$$\Psi_b = 0$$

$$\Psi_c$$

$$d\mathbf{s} = \rho d\phi dz \underline{\hat{\rho}}$$

$$0 < z < 1$$

$$\phi = 0 < \phi < 2\pi$$

$$\psi_s = \int (\rho^2 \cos^2 \phi \, \underline{\hat{a}}_\rho) \cdot (\rho \, d\phi \, dz \, \underline{\hat{a}}_\rho)$$

$$\psi_s = \int \rho^3 \cos^2 \phi \, d\phi \, dz$$

$$\psi_s = \rho^3 \int_0^{2\pi} \underbrace{\cos^2 \phi \, d\phi}_{\pi} \int_0^1 dz$$

$$= 4^3 (\pi) (1) = \boxed{64\pi}$$

$$\psi = \int_V \nabla \cdot D \, dV$$

$$\nabla \cdot D = \frac{1}{\rho} \frac{\partial D}{\partial \rho} \rho^3 \cos^2 \phi + \frac{1}{\rho} \frac{\partial}{\partial \phi} z \sin \phi$$

$$\nabla \cdot D = 3\rho \cos^2 \phi + \frac{z}{\rho} \cos \phi$$

$$\psi = \int (3\rho \cos^2 \phi + \frac{z}{\rho} \cos \phi) \rho \, d\rho \, d\phi \, dz$$

$$\psi = \int 3\rho^2 \cos^2 \phi \, d\rho \, d\phi \, dz + z \cos \phi \, d\rho \, d\phi \, dz$$

$$\psi = 3 \int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2 \varphi d\varphi \int_0^1 dz + \int_0^1 z dz$$

$$\int_0^{2\pi} \cos \varphi d\varphi \int_0^4 d\rho$$

zero

$$= 3 \left[\frac{\rho^3}{3} \right]_0^4 \left[\pi \right] \left[1 \right] + \left[\frac{z^2}{2} \right]_0^1 \left[\sin \varphi \right]_0^{2\pi} \left[4 \right]$$

$$3 \left(\frac{64}{3} \right) \pi = \boxed{64\pi}$$

PRACTICE EXERCISE 3.8

Determine the curl of the vector fields in Practice Exercise 3.6 and evaluate them at the specified points.

a)

$$A = yz \mathbf{a}_x + 4xy \mathbf{a}_y + y \mathbf{a}_z \quad (1, -2, 3)$$

$$\nabla \times A = \begin{vmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 4xy & y \end{vmatrix}$$

$$\left(\frac{\partial y}{\partial y} - \frac{\partial 4xy}{\partial z} \right) \mathbf{a}_x - \left(\frac{\partial y}{\partial x} - \frac{\partial yz}{\partial z} \right) \mathbf{a}_y + \left(\frac{\partial 4xy}{\partial x} - \frac{\partial yz}{\partial y} \right) \mathbf{a}_z$$

$$(1-0) \mathbf{a}_x - (0-y) \mathbf{a}_y + (4y-z) \mathbf{a}_z$$

$$\nabla \times A = \hat{a}_x + y\hat{a}_y + (4y-z)\hat{a}_z$$

at $(1, -2, 3)$

$$\nabla \times A = \hat{a}_x - 2\hat{a}_y - 11\hat{a}_z$$

b) $B = \rho z \sin \phi \hat{a}_\rho + 3\rho z^2 \cos \phi \hat{a}_\phi$ $(5, \frac{\pi}{2}, 1)$

$$\nabla \times B = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho z \sin \phi & 3\rho^2 z^2 \cos \phi & 0 \end{vmatrix}$$

$$\nabla \times B = \frac{1}{\rho} \left\{ \left(\frac{\partial}{\partial \phi} 0 - \frac{\partial}{\partial z} 3\rho^2 z^2 \cos \phi \right) \hat{a}_\rho - \left(\frac{\partial}{\partial \rho} 0 - \frac{\partial}{\partial z} \rho z \sin \phi \right) \rho \hat{a}_\phi + \left(\frac{\partial}{\partial \rho} 3\rho^2 z^2 \cos \phi - \frac{\partial}{\partial \phi} \rho z \sin \phi \right) \hat{a}_z \right\}$$

$$\nabla \times B = \frac{1}{\rho} \left\{ -6\rho^2 z \cos \phi \hat{a}_\rho + \rho^2 \sin \phi \hat{a}_\phi + (6\rho z^2 \cos \phi + \rho z \cos \phi) \hat{a}_z \right\}$$

$$\nabla \times B = -6\rho z \cos \phi \hat{a}_\rho + \rho \sin \phi \hat{a}_\phi + (6z - 1) z \cos \phi \hat{a}_z$$

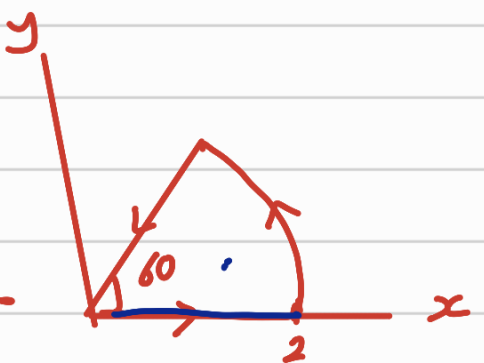
$$\nabla \times B = 5(\sin \frac{\pi}{2}) \hat{a}_\phi = 5 \hat{a}_\phi$$

PRACTICE EXERCISE 3.9

Use Stokes's theorem to confirm your result in Practice Exercise 3.2.

$$\oint A \cdot d\mathbf{Q} = \int_S \nabla \times A \cdot d\mathbf{s}$$

على من، لعل باستخدام لثابت



$$\nabla \times A = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho \cos \phi & \rho(0) & z \sin \phi \end{vmatrix}$$

$$\nabla \times A = \frac{1}{\rho} [z \cos \phi - 0] \hat{a}_\rho - \frac{1}{\rho} [0 - 0] \rho \hat{a}_\phi$$

$$\frac{1}{\rho} [0 - -\rho \sin \phi] \hat{a}_z$$

$$\nabla \times A = \frac{1}{\rho} [z \cos \phi \hat{a}_\rho] + [\sin \phi \hat{a}_z]$$

$$\oint (\nabla \times A) \cdot d\mathbf{s} = \int_S \left[\frac{z \cos \phi \hat{a}_\rho + \sin \phi \hat{a}_z \right] \cdot \left[\rho d\rho d\phi \hat{a}_z \right]$$

Zero

$\hat{a}_z \cdot \hat{a}_z = 1$

$$\int (\nabla \times A) \cdot d\mathbf{s} = \int_0^5 \int_0^{60} \rho \sin \phi \, d\rho \, d\phi$$

$$= \int_0^2 \rho \, d\rho \int_0^{60} \sin \phi \, d\phi$$

$$= \left[\frac{\rho^2}{2} \right]_0^2 \left[-\cos \phi \right]_0^{60}$$

$$\left(\frac{4-0}{2} \right) (-\cos 60 - \cos 0)$$

$$2 \left(-\frac{1}{2} + 1 \right)$$

$$2 \left(\frac{1}{2} \right) = 1$$

PRACTICE EXERCISE 3.10

For a scalar field V , show that $\nabla \times \underline{\nabla V} = 0$; that is, the curl of the gradient of any scalar field vanishes.

(A_x, A_y, A_z)

Answer: Proof.

$$\underline{\nabla} V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z$$

$$\nabla \times \underline{\nabla} V = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y} \right] \hat{a}_x - \left[\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x} \right] \hat{a}_y$$

$$+ \left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \right] \hat{a}_z$$

$$= 0 \hat{a}_x + 0 \hat{a}_y + 0 \hat{a}_z$$

$$= 0$$