Electromagnetic Field PHYS 321

HW#1

2nd semester 1446

Q1:

Example 1.1

If $A = 10a_x - 4a_y + 6a_z$ and $B = 2a_x + a_y$, find:

(a) the component of A along a_y $= -4$

(b) the magnitude of 3A - B, (c) a unit vector along A + 2B.

b) $3A - B$

$3(10a_x - 4a_y + 6a_z) - (2a_x + a_y)$

$30a_x - 12a_y + 18a_z - 2a_x - a_y$

$28a_x - 13a_y + 18a_z$

$= \sqrt{28^2 + 13^2 + 18^2} = 35.74$

c) $a_c = \dfrac{C}{|C|}$ $\qquad C = A + 2B$

$C = 10a_x - 4a_y + 6a_z + 4a_x + 2a_y$

$C = 14a_x - 2a_y + 6a_z$

$$\hat{a}_c = \frac{14a_x - 2a_y + 6a_z}{\sqrt{14^2 + 2^2 + 6^2}} = 0.91a_x - 0.13a_y + 0.39a_z$$

Q2:

PRACTICE EXERCISE 1.5

Let $E = 3a_y + 4a_z$, **and** $F = 4a_x - 10a_y + 5a_z$

(a) Find the component of E along F.

(b) Determine a unit vector perpendicular to both E and F.

Answer: (a) (-0.2837, 0.7092, -0.3546), (b) ± (0.9398, 0.2734, -0.205).

a) Component of E along F $= \dfrac{(F \cdot E)}{|F|^2} \vec{F}$

$$= \frac{[0 - 30 + 20]}{4^2 + 10^2 + 5^2} (4a_x - 10a_y + 5a_z)$$

$$= \frac{-10}{141} (4a_x - 10a_y + 5a_z)$$

$$E_F = -6.2887a_x + 0.7092a_y - 0.3546a_z$$

b) $\dfrac{E \times F}{|E \times F|} = \dfrac{55a_x + 16a_y - 12a_z}{58.52} = 0.939a_x + 0.273a_y - 0.205a_z$

$$E \times F = \begin{vmatrix} a_x & a_y & a_z \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = 55a_x + 16a_y - 12a_z$$

$$|E \times F| = \sqrt{55^2 + 16^2 + 12^2} = 58.52$$

Q3:

Review problem

1.7 Let $\mathbf{F} = 2\mathbf{a}_x - 6\mathbf{a}_y + 10\mathbf{a}_z$ and $\mathbf{G} = \mathbf{a}_x + G_y\mathbf{a}_y + 5\mathbf{a}_z$. If $\mathbf{F}$ and $\mathbf{G}$ have the same unit vector, G_y is

(a) 6 (d) 0

(b) −3 (e) 6

1.9 The component of $\underset{A}{6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z}$ along $\underset{B}{3\mathbf{a}_x - 4\mathbf{a}_y}$ is

(a) $-12\mathbf{a}_x - 9\mathbf{a}_y - 3\mathbf{a}_z$

(b) $30\mathbf{a}_x - 40\mathbf{a}_y$

(c) 10/7

(d) 2

(e) 10

$$\frac{A\cdot B}{|B|} = \frac{18-8}{5} = \frac{10}{5} = 2$$

1.10 Given $\mathbf{A} = -6\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$, the projection of $\mathbf{A}$ along $\mathbf{a}_y$ is

(a) −12

(b) −4

(c) 3

(d) 7

(e) 12

1.7)
$$\hat{F} = \frac{2a_x - 6a_y + 10a_z}{\sqrt{2^2+6^2+10^2}} = \frac{2a_x}{\sqrt{140}} - \frac{6a_y}{\sqrt{140}} + \frac{10a_z}{\sqrt{140}}$$

$$\hat{G} = \frac{a_x + G_y a_y + 5a_z}{\sqrt{1^2+G^2+5^2}} = \frac{a_x}{\sqrt{26+G^2}} + \frac{G_y a_y}{\sqrt{26+G^2}} + \frac{5a_z}{\sqrt{26+G^2}}$$

$$\frac{2}{\sqrt{140}} = \frac{1}{\sqrt{26+G^2}} \Rightarrow \sqrt{140} = 2\sqrt{26+G^2}$$

$$140 = 4(26+G^2) \Rightarrow G^2 = \sqrt{\frac{140}{4} - 26} = \sqrt{9} \begin{cases} \to +3 \\ \to -3 \end{cases}$$

Problems

$(0, 1, -3)$

Q4:

1.22 **E** and **F** are vector fields given by $\mathbf{E} = 2x\mathbf{a}_x + \mathbf{a}_y + yz\mathbf{a}_z$ and $\mathbf{F} = xy\mathbf{a}_x - y^2\mathbf{a}_y + xyz\mathbf{a}_z$. Determine:

(a) $|\mathbf{E}|$ at (1, 2, 3)

(b) The component of **E** along **F** at (1, 2, 3) $F = 2a_x - 4a_y + 6a_z$

(c) A vector perpendicular to both **E** and **F** at (0, 1, −3) whose magnitude is unity

a) $E = 2a_x + a_y + 6a_z$

$|E| = \sqrt{1 + 2^2 + 6^2} = \sqrt{41}$

b) $\dfrac{E \cdot F}{|F|^2} \cdot \bar{F} = \dfrac{36}{2^2 + 4^2 + 36}(2a_x - 4a_y + 6a_z)$

$\dfrac{36}{56}(2a_x - 4a_y + 6a_z)$

$= 1.28a_x - 2.571a_y + 3.85a_z$

c) $\dfrac{E \times F}{|E \times F|} = \dfrac{-3a_x}{3} = \hat{a}_x$

$E \times F \begin{vmatrix} a_x & a_y & a_z \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = -3a_x + 0a_y + 0a_z$

$|E \times F| = \sqrt{3^2} = 3$

Q5:

EXAMPLE 2.3 Two uniform vector fields are given by $\mathbf{E} = -5\mathbf{a}_\rho + 10\mathbf{a}_\phi + 3\mathbf{a}_z$ and $\mathbf{F} = \mathbf{a}_\rho + 2\mathbf{a}_\phi - 6\mathbf{a}_z$. Calculate

(a) $|\mathbf{E} \times \mathbf{F}|$

(b) The vector component of $\mathbf{E}$ at $P(5, \pi/2, 3)$ parallel to the line $x = 2, z = 3$

(c) The angle $\mathbf{E}$ makes with the surface $z = 3$ at P

ϕ

$$E \times F = \begin{vmatrix} a_x & a_y & a_z \\ -5 & 10 & 3 \\ 1 & 2 & -6 \end{vmatrix} = -66a_x - 27a_y - 20a_z$$

$$|E \times F| = \sqrt{66^2 + 27^2 + 20^2} = 74.06$$

$$A_x = \cos\phi\, A_\rho - \sin\phi\, A_\phi$$

$$A_y = \sin\rho\, A_\rho + \cos\phi\, A_\phi$$

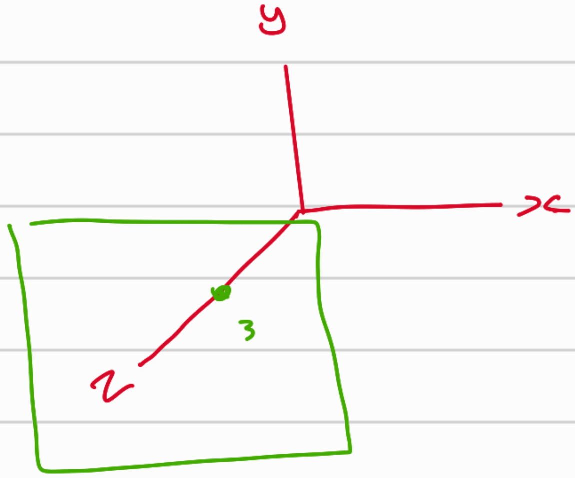
$$A_z = A_z$$

b)

$$A_y = \sin\frac{\pi}{2}(-5) + \cos \pi/2\ (10)$$

zer

$$\boxed{A_y = -5}$$



c) $E \cdot a_z = |E|\,|\hat{a}_z| \cos\theta$

$$\cos\theta = \frac{E \cdot a}{|E|}$$

$$\cos\theta = \frac{(\cancel{E_x a_x} + \cancel{E_y a_y} + E_z a_z) \cdot a_z}{|E|} = \frac{3}{\sqrt{5^2 + 10^2 + 3^2}}$$

$$\cos\theta = \frac{3}{\sqrt{134}} \qquad \theta = \cos^{-1}\left(\frac{3}{\sqrt{134}}\right) = 74.98$$

Q6:

PRACTICE EXERCISE 2.3

Given the vector field

$$\mathbf{H} = \rho z \cos\phi\, \mathbf{a}_\rho + e^{-2} \sin\frac{\phi}{2}\, \mathbf{a}_\phi + \rho^2 \mathbf{a}_z$$

At point $(1, \pi/3, 0)$, find $H = 0\, a_\rho + e^{-2} \sin\frac{\pi}{6}\, a_\phi + a_z$

$H = \frac{1}{2} e^{-2} a_\phi + a_z$

(a) $\mathbf{H} \cdot \mathbf{a}_x$

(b) $\mathbf{H} \times \mathbf{a}_\theta$

(c) The vector component of **H** normal to surface $\rho = 1$ a_ρ

(d) The scalar component of **H** tangential to the plane $z = 0$

Answer: (a) -0.433, (b) $-0.5\, \mathbf{a}_\rho$, (c) $0\, \mathbf{a}_\rho$, (d) 0.5.

a) $a_x = \cos\phi\, a_\rho - \sin\phi\, a_\phi$

$$H \cdot a_x = \left(\frac{e^{-2}}{2} a_\phi + a_z\right) \cdot \left(\cos\frac{\pi}{3} a_\rho - \sin\frac{\pi}{3} a_\phi\right)$$

$$= -\frac{e^{-2}}{2} \sin\frac{\pi}{3} = \frac{e^{-2}}{2} \frac{\sqrt{3}}{2} =$$

$$= -0.6586$$

b) $H \times a_\theta$

$$\boxed{a_\theta = \cos\theta\, a_\rho - \sin\theta\, a_z}$$

$$H \times a_\theta = \begin{array}{ccc} a_\rho & a_\phi & a_z \end{array} \begin{vmatrix} 0 & \frac{e^{-2}}{2} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{\rho}{z}\right)}$$

$$= \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$$

$$a_\theta = 0\, a_\rho - 1\, a_z$$

$$\boxed{a_\theta = -a_z}$$

$$= \frac{e^{-2}}{2} a_\rho = -0.067\, a_\rho$$

c) $H \cdot a_\rho$

$$\left(\frac{e^{-2}}{2} a_\phi + a_z \right) \cdot 1 a_\rho = 0$$

d) المركبة المماسية لـ مستوى $z=0$ تعني
المركبات الاخرى ρ ϕ

$$H = \frac{e^{-2}}{2} e \, a_\phi + a_z$$

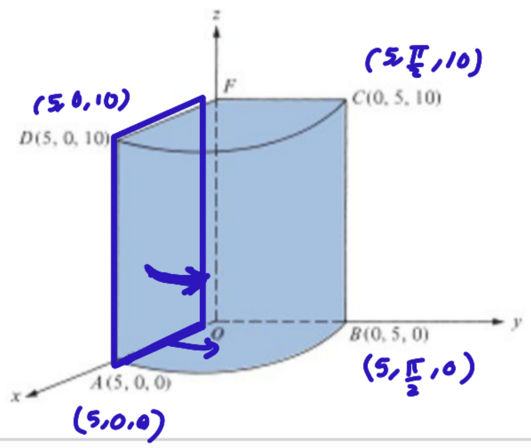
اتجاه فقط ϕ

$$= \sqrt{\left(\frac{e^{-2}}{2}\right)^2} = \frac{e^{-2}}{2}$$

Q7:

EXAMPLE 3.1 Consider the object shown in Figure 3.7. Calculate

(a) The distance BC
(b) The distance CD
(c) The surface area $ABCD$
(d) The surface area ABO
(e) The surface area $AOFD$ 50
(f) The volume $ABDCFO$



a) $BC = \int_0^{10} dz = z\Big]_0^{10} = [10-0] = 10$

$$dr = \cancel{d\rho\, \hat{a}_\rho} + \rho\, d\varphi\, \hat{a}_\varphi + \cancel{dz\, \hat{a}_z}$$

b) CD

$$l = \int dl = \int_0^{\pi/2} \rho\, d\varphi = 5\ \varphi\Big]_0^{\pi/2} = 5 \times \frac{\pi}{2} = 2.5\pi$$

$ds = da$
$\rho\, d\varphi\, dz\ \hat{a}_\rho$
$d\rho\, dz\ \hat{a}_\varphi$
$\rho\, d\varphi\, d\rho\ \hat{a}_z$

c)

$$da = \int \rho\, d\varphi\, dz$$

$$5 \int_0^{2\pi} d\varphi \int_0^{10} dz = 5 \times \frac{\pi}{2} \times 10 = 25\pi$$

d) $\int \rho\, d\varphi\, d\rho = \int_0^5 \rho\, d\rho \int_0^{\pi/2} d\varphi$

$$\frac{\rho^2}{2}\Big]_0^5 \times \frac{\pi}{2} = \quad 6.25\pi$$

e) $da = d\rho \, dz$

$$a = \int d\rho \, dz = \int_0^5 d\rho \int_0^{10} dz = 5 \times 10 = 50$$

f) $v = \int dv = \int \rho \, d\rho \, d\phi \, dz$

$$\int_0^5 \rho \, d\rho \int_0^{\pi/2} d\phi \int_0^{10} dz$$

$$\frac{5^2}{2} \times \frac{\pi}{2} \times 10 = 62.5\pi$$

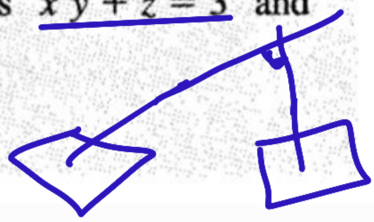
Q12:

PRACTICE EXERCISE 3.5

f

Calculate the angle between the normals to the surfaces $x^2y + z = 3$ and
g $x \log z - y^2 = -4$ at the point of intersection $(-1, 2, 1)$.

Answer: 73.4°.



$$\nabla f = \frac{\partial f}{\partial x} a_x + \frac{\partial f}{\partial y} a_y + \frac{\partial f}{\partial z} a_z = 2xy\,\hat{a}_x + x^2\hat{a}_y + \hat{a}_z$$

$$\nabla g = \frac{\partial g}{\partial x}\hat{a}_x + \frac{\partial g}{\partial y}\hat{a}_y + \frac{\partial g}{\partial z}\hat{a}_z = \log z\,\hat{a}_x - 2y\,\hat{a}_y + \frac{x}{z}\hat{a}_z$$

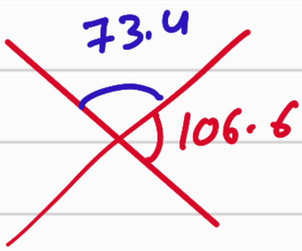
at $(-1, 2, 1)$

$$\nabla f = -4\hat{a}_x + \hat{a}_y + \hat{a}_z$$

$$\nabla g = -4\hat{a}_y - a_z$$

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla g|\,|\nabla f|} = \frac{(-4\times 0) + (-4) + (-1)}{\sqrt{16+1+1}\,\sqrt{16+1}}$$

$$\theta = \cos^{-1}\left[\frac{-5}{\sqrt{18}\sqrt{17}}\right] = 106.6^\circ$$



Q13:

EXAMPLE 3.6

Determine the divergence of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho\sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \frac{1}{r^2}\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

c) $\nabla\cdot T = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial T_\phi}{\partial\phi}$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\cancel{r^2}\frac{1}{\cancel{r^2}}\cos\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} r\sin^2\theta\cos\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\cos\theta$

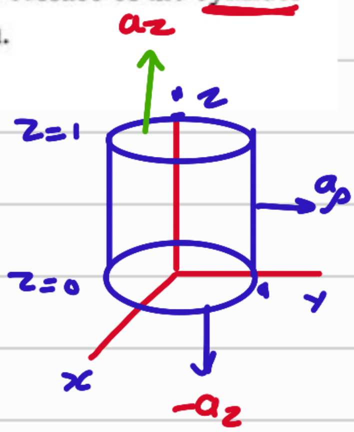
$0 + \frac{1}{\cancel{r\sin\theta}}\cancel{r}\cos\phi\, 2\cancel{\sin\theta}\cos\theta + 0$

$2\cos\phi\cos\theta$

Q14:

EXAMPLE 3.7

If $\mathbf{G}(r) = 10e^{-2z}(\rho\mathbf{a}_\rho + \mathbf{a}_z)$, determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1, 0 \le z \le 1$. Confirm the result using the divergence theorem.



$\Psi = \oint G\cdot ds = \int_v (\nabla\cdot G)\, dv$

$\Psi = \Psi_t + \Psi_b + \Psi_s$

$\Psi_t = 10e^{-2z}\int(\rho\hat{a}_\rho + a_z)\cdot(\rho\, d\rho\, d\phi\, \hat{a}_z)$

$= 10e^{-2}\int_0^1\int_0^{2\pi}\rho\, d\rho\, d\phi = 10e^{-2}\frac{\rho^2}{2}\Big]_0^1\ \phi\Big]_0^{2\pi} = 10e^{-2}$

$$\Psi_b = +10 e^{-2z} \iint (\rho \cancel{\hat{a}_\rho} + a_z)(\rho\, d\rho\, d\phi(-a_z))$$

$$\Psi_b = -10 e^{0} \int_0^1 \int_0^{2\pi} \rho\, d\rho\, d\phi = -10 \left.\frac{\rho^2}{2}\right]_0^1 \left.\phi\right]_0^{2\pi} = -10\pi$$

$$\Psi_s = 10 \int e^{-2z}(\rho \hat{a}_\rho + a_z)\cdot(\rho\, dz\, d\phi\, a_\rho)$$

$$10 \int_0^1 \int_0^{2\pi} e^{-2z} \rho^2\, dz\, d\phi = 10 \int_0^1 e^{-2z} dz \int_0^{2\pi} d\phi$$

$$10 \left.\frac{e^{-2z}}{-2}\right]_0^1 \left.\phi\right]_0^{2\pi} = \frac{10}{-2}\left[e^{-2} - e^{0}\right] 2\pi$$

$$\Psi_s = -10\pi\left[e^{-2} - 1\right] = 10\pi(1 - e^{-2})$$

$$\Psi = \Psi_t + \Psi_b + \Psi_s = 10\pi e^{-2} - 10\pi + 10\pi(1 - e^{-2}) = 0$$

$$\int (\nabla \cdot G)\, dv = \int 0\, dv = 0$$

$$\nabla \cdot G = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho G_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi} G_\phi + \frac{\partial}{\partial z} G_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho^2(10e^{-2z}) + \frac{\partial}{\partial z} 10e^{-2z}$$

$$= 20e^{-2z} - 20e^{-2z} = 0$$

Q15:

Probl.3.14(P105). Determine the unit vector normal to $S(x,y,z) = x^2 + y^2 - z$ at point $(1,3,0)$.

$$\hat{S}_\perp = \frac{\nabla S}{|\nabla S|} = \frac{\partial S}{\partial x} a_x + \frac{\partial S}{\partial y} a_y + \frac{\partial S}{\partial z} a_z$$

$$\nabla S = 2x\,\hat{a}_x + 2y\,\hat{a}_y - \hat{a}_z$$

$$\nabla S = 2\hat{a}_x + 6\hat{a}_y - a_z$$

$$\hat{S}_\perp = \frac{\nabla S}{|\nabla S|} = \frac{2\hat{a}_x + 6a_y - a_z}{\sqrt{4+36+1}}$$

$$= \frac{2}{\sqrt{41}} a_x + \frac{6}{\sqrt{41}} a_y - \frac{1}{\sqrt{41}} \hat{a}_z$$

Q16:

Prob. (3.15, p105) The temperature in auditorium is given by $T = x^2 + y^2 - z$. A mosquito located at (1,1,2) in the auditorium desires to fly in such a direction that it will get warm as soon as possible. In what direction must it fly?

$$\hat{T}_\perp = \frac{\nabla T}{|\nabla T|} = \frac{2a_x + 2a_y - a_z}{\sqrt{9}} = \frac{2}{3}\hat{a}_x + \frac{2}{3}\hat{a}_y - \frac{1}{3}\hat{a}_z$$

$$\nabla T = \frac{\partial T}{\partial x}\hat{a}_x + \frac{\partial T}{\partial y}\hat{a}_y + \frac{\partial T}{\partial z}\hat{a}_z$$

$$= 2x\,a_x + 2y\,a_y - a_z = 2\hat{a}_x + 2\hat{a}_y - \hat{a}_z$$

Q17:

EXAMPLE 3.12 Show that the vector field **A** is conservative if **A** possesses one of these two properties:

(a) The line integral of the tangential component of **A** along a path extending from a point P to a point Q is independent of the path.

(b) The line integral of the tangential component of **A** around any closed path is zero.

$$\nabla \times A = 0 \iff A \text{ is conservative}$$

ازا اثبتت A محققة ازا

a)

$$A = -\nabla V = -\left[\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} \bar{a}_z\right]$$

$$\int_P^Q A \cdot dl = -\int_P^Q \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$-\int_P^Q \left[\frac{\partial V}{\cancel{\partial x}} \frac{\cancel{dx}}{ds} + \frac{\partial V}{\cancel{\partial y}} \frac{\cancel{dy}}{ds} + \frac{\partial V}{\cancel{\partial z}} \frac{\cancel{dz}}{ds}\right] ds$$

$$-\int_P^Q \frac{dV}{ds} ds = -\int_P^Q dV$$

$$-V\Big]_P^Q = -\left[V(Q) - V(P)\right]$$

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b)

$$\oint A\, dl = -\int_P^P dV = V(P) - V(P) = 0$$

Q19:

EXAMPLE 3.11 Find the Laplacian of the scalar fields of Example 3.3; that is,

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2 \theta \cos \phi$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial W}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial W}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 W}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 (10 \sin^2\theta \cos\phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta} \sin\theta \times 20r \cos\theta \cos\phi$$

$$+ \frac{-1}{r^2 \sin^2\theta} 10 r \sin^2\theta \cos\phi$$

$$= \frac{20 r \sin^2\theta \cos\phi}{r^2} + \frac{20 r \cos\phi}{r^2 \sin\theta}\left[2\sin\theta \cos^2\theta - \sin^3\theta\right]$$

$$- \frac{10}{r}\cos\phi$$

$$= \frac{20}{r}\sin^2\theta \cos\phi + \frac{20\cos\phi}{r}\left[\cos^2\theta - \sin^2\theta\right] \quad \cos 2\theta$$

$$- \frac{10}{r}\cos\phi$$

$$\frac{10\cos\phi}{r}\left[2\sin^2\theta + 2\cos 2\theta - 1\right]$$