2. First Order Differential Equations

2.1 Introduction

In this chapter we discuss methods to find the solution of the first order differential equation.

The general form of the first order differential equation is

$$\frac{dy}{dx} = f(x,y) \quad \text{or} \quad M(x,y)dx + N(x,y)dy = 0 \checkmark$$

$$\frac{dy}{dx} = 2xy \qquad\qquad 2x\,dx + xy\,dy = 0$$

2.2 Separable Variables

فصل المتغيرات

Definition 2.2.1 A first order differential equation of the form $\frac{dy}{dx} = f(x)g(y)$ where $f(x)$ and $g(y)$ are functions of x and y respectively, is called separable variable equation.

Rearranging this equation we obtain

$$\frac{dy}{g(y)} = f(x)dx \Longrightarrow \int g(y)dy = \int f(x)dx$$

خطوات طريقه فصل المتغيرات

① نفصل قيم x , dx عن قيم dy y

$$y\,dy = x\,dx$$

② نقوم بجعله تكامل للطرفين

③ نضف الثابت c ال احد الطرفين ونرتب كمعادلة.

④ في حال وجود initial value نحسب قيمة c ونعوضها

Example 2.1 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$

$$\frac{dy}{dx} = \frac{x}{y}$$

① فصل X عن Y

$$y\,dy = x\,dx$$

② تكامل الطرفين

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

③ نضبط C وترتيب المعادلة

$$\cancel{2}\,\frac{y^2}{\cancel{2}} = \cancel{2}\,\frac{x^2}{\cancel{2}} + 2C \quad ✓$$

$$\boxed{y^2 = x^2 + 2C}$$

$$y^2 - x^2 = 2C \qquad y = \sqrt{x^2 + 2C}$$

$$2C \Rightarrow C_1 \qquad A$$

$$y^2 = x^2 + C_1$$

قيمه ابتدائيه

■ **Example 2.2** Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = 3$

$x \quad y$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\int y\, dy = \int -x\, dx$$

$$\cancel{2}\frac{y^2}{\cancel{2}} = \cancel{2} - \frac{x^2}{\cancel{2}} + c$$

نضرب ب 2

$$y^2 = -x^2 + 2c$$

$y(4) = 3$

$x \quad y$

نعوض القيمه الابتدائيه $\boxed{\text{خطوه } 4}$

$$3^2 = -(4^2) + 2c$$

$$9 = -16 + 2c$$

$$2c = 9 + 16$$

$$\boxed{2c = 25}$$

$$y^2 = -x^2 + \cancel{2c}$$

$$\boxed{y^2 = -x^2 + 25}$$

$$y^2 + x^2 = 25 \quad ✓$$

■ **Example 2.3** Solve the differential equation $\frac{dy}{dx} = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{y}{x} \qquad \frac{\cancel{x}\,dy}{y\,\cancel{x}} = \frac{\cancel{y}\,dx}{\cancel{y}\,x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x} \implies \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln y = \ln x + \ln c$$

$$\ln y - \ln x = \ln c$$

$$\ln \frac{y}{x} = \ln c$$

$$\frac{y}{x} = c$$

$$\boxed{y = cx}$$

■ **Example 2.4** Solve the differential equation $\frac{dy}{dx} = \cos 5x$

$$\frac{dy}{dx} = \cos 5x$$

$$\int dy = \int \cos 5x \quad dx$$

$$y = \frac{\sin 5x}{5} + C$$

$$\boxed{y = \frac{1}{5}\sin 5x + C}$$

$$\int e^{x} = e^{x}$$

$$\int e^{3x} = \frac{e^{3x}}{3}$$

$$\int \cos x = \sin x$$

$$\int \cos 3x = \frac{\sin 3x}{3}$$

■ **Example 2.5** Solve the differential equation $(1+x)dy - ydx = 0$

$$(1+x)\,dy - y\,dx = 0$$

$$\frac{\cancel{(1+x)}\,dy}{y\,\cancel{(1+x)}} = \frac{\cancel{y}\,dx}{(1+x)\cancel{y}}$$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln y = \ln(1+x) + c$$

$$e^{\ln y} = e^{\ln(1+x) + c}$$

$$y = \cancel{e}^{\cancel{\ln}(1+x)} \cdot e^{c}$$

$$y = (1+x) \cdot A$$

$$y = A(1+x)$$

$\int \frac{1}{x}\,dx = \ln x$

$\int \frac{1}{2x+2} = \frac{\ln(2x+2)}{2}$

$\int \frac{dx}{1+x} = \ln|1+x|$

[illegible]

■ **Example 2.6** Solve the differential equation $x\dfrac{dy}{dx} - y = 2x^2y$

$$x\frac{dy}{dx} - y = 2x^2y$$

$$x\frac{dy}{dx} = 2x^2y + y$$

$$x\frac{dy}{dx} = y(2x^2+1)$$

$$\frac{\cancel{x}\,dy}{\cancel{x}\,y} = \frac{\cancel{y}(2x^2+1)\,dx}{x\,\cancel{y}}$$

$$\int\frac{dy}{y} = \int\frac{(2x^2+1)}{x}\,dx$$

$$\ln y = \int 2x\,dx + \int\frac{1}{x}\,dx$$

$$\ln y = \frac{\cancel{2}x^2}{\cancel{2}} + \ln x + c$$

$$\boxed{\ln y = x^2 + \ln x + c}$$

✓

$$e^{\ln y} = e^{x^2+\ln x+c}$$

$$y = e^{x^2}\cdot \underbrace{e^{\ln x}}_{x}\cdot\underbrace{e^{c}}_{c_1} = c_1 x e^{x^2}$$

■ **Example 2.7** Solve the differential equation $y' = e^{2x}$

$$y' = \frac{dy}{dx}$$

$$y' = e^{2x}$$

$$\frac{dy}{dx} = e^{2x}$$

$$\int dy = \int e^{2x}\,dx$$

$$y = \frac{e^{2x}}{2} + c$$

$$\boxed{y = \frac{1}{2}e^{2x} + c}$$

2.3 Homogeneous متجانسة

Homogeneous equation such type of equation can be reduced to variable separable from by the substitution as discusses below.

Definition 2.3.1 A function $f(x,y)$ is a homogeneous function of its arguments of degree n

$$f(\lambda x, \lambda y) = \lambda^n f(x,y), \qquad n \in \mathbb{R}$$

A first order differential equation, $\boxed{M(x,y)dx + N(x,y)dy = 0}$ is called homogeneous if both coefficients $M(x,y)dx$ and $N(x,y)dy$ are homogeneous functions of the same degree.

$$M(\lambda x, \lambda y) = \lambda^n M(x,y) \qquad \text{and} \qquad N(\lambda x, \lambda y) = \lambda^n N(x,y)$$

الداله M و الداله N في المعادلة $M\,dx + N\,dy = 0$ تكون لهما نفس الدرجة

A homogeneous differential equation can always be expressed in the form

$$\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

درجة M هي نفس درجة N

امثلة على معادلات Homo

$y'dx + x'dy = 0$ Homo

$\frac{dy}{dx} = \frac{y^2}{xy} \Rightarrow (xy)\,dy = (y^2)\,dx \Rightarrow y^2dx - xy\,dy = 0$ Homo

$\frac{dy}{dx} = \frac{x^2+y^2}{xy} \Rightarrow xy\,dy = (x^2+y^2)\,dx$ Homo

$\underset{2}{(x^2+y^2)}\,dx - \underset{2}{(xy)}\,dy = 0$

Method of Solution for Homogeneous Equations:

A homogeneous differential equation can be solved by substituting

We introduce a new unknown function $u = \frac{y}{x}, \Longrightarrow y = ux$

$$dy = u\,dx + x\,du \qquad \text{or} \qquad \frac{dy}{dx} = u + x\frac{du}{dx}$$

خطوات حل المعادلة المتجانسة

① بعد التأكد ان المعادله هي Homo نفرض $u = \frac{y}{x}$

$$\boxed{y = ux \qquad dy = u\,dx + x\,du}$$

② بعد التعويض نرتب المعادله بحيث نحصل فقط على متغيرين u x

③ نطبق الحل بطريقة فصل المتغيرات u x

④ نرجع u بدلاله y $\boxed{u = \frac{y}{x}}$

■ **Example 2.8** Solve the differential equation $(\overset{M}{x-y})dx+\overset{N}{x}dy=0$

① نتأكد ان المعادلة Homo ✓ اعلى درجه في M و N هي 1

② نفرض $dy = u\,dx + x\,du$ $\quad y = ux$

$$(x-y)\,dx + x\,dy = 0$$
$$(x-ux)\,dx + x(u\,dx + x\,du) = 0$$
$$x\,dx - \cancel{ux\,dx} + \cancel{xu\,dx} + x^2du = 0$$
$$x\,dx + x^2du = 0$$

③ فصل المتغيرات

$$\frac{x\,dx}{x^2} = -\frac{\cancel{x^2}\,du}{\cancel{x^2}}$$
$$\int \frac{1}{x}\,dx = -\int du$$
$$\ln x = -u + c$$

④ نرجع y $\qquad u = \frac{y}{x}$

$$\ln x = -\frac{y}{x} + c$$

نضرب المعادلة ب x

$$x\ln x = -y + cx$$

$$\boxed{y = -x\ln x + cx}$$

■ **Example 2.9** Solve the differential equation $(\underset{2}{y^2}+\underset{2}{xy})dx+\underset{2}{x^2}dy=0$

M N

1) this equ is Homogenous

2) $y=ux \qquad dy=u\,dx+x\,du$ تعويض

$$(y^2+xy)dx+x^2dy=0$$

$$(u^2x^2+x^2u)dx+x^2(u\,dx+x\,du)=0$$

3) $u^2x^2dx+x^2u\,dx+x^2u\,dx+x^3du=0$

$$u^2x^2dx+2x^2u\,dx+x^3du=0$$

$$\frac{x^2\cancel{(u^2+2u)}dx}{x^3\cancel{(u^2+2u)}}=\frac{-\cancel{x^3}du}{\cancel{x^3}(u^2+2u)}$$

$$\int -\frac{1}{x}dx=\int\frac{du}{u(u+2)}$$

$$-\ln x=\int\frac{1/2}{u}+\frac{-1/2}{(u+2)}du$$

$$\frac{1}{u(u+2)}=\frac{\cancel{A}^{\,1/2}}{u}+\frac{\cancel{B}^{\,-1/2}}{u+2}$$

$$\frac{1}{u\cancel{(u+2)}}=\frac{A(u+2)+Bu}{u\cancel{(u+2)}}$$

$$1=Au+2A+Bu$$

$$1=(A+B)u+2A$$

$A+B=0 \qquad 2A=1 \qquad A=\frac{1}{2}$

$A=-B \qquad B=-\frac{1}{2}$

$$-\ln x=\frac{1}{2}\int\frac{1}{u}du-\frac{1}{2}\int\frac{1}{u+2}du$$

$$-\ln x=\frac{1}{2}\ln u-\frac{1}{2}\ln(u+2)+C$$

$$-\ln x=\frac{1}{2}\ln\frac{y}{x}-\frac{1}{2}\ln\left(\frac{y}{x}+2\right)+C$$

■ **Example 2.10** Solve the differential equation $(x-y)dx+(x+y)dy=0$

1) this equ is Homo

2) $y = ux \qquad dy = u\,dx + x\,du$

$$(x-y)\,dx + (x+y)\,dy = 0$$
$$(x-ux)\,dx + (x+ux)(u\,dx + x\,du) = 0$$

$$x\,dx - \cancel{ux\,dx} + \cancel{xu\,dx} + x^2du + u^2x\,dx + ux^2du = 0$$

$$x\,dx + x^2du + u^2x\,dx + ux^2du = 0$$

$$x\,dx + u^2x\,dx = -x^2du - ux^2du$$

$$\frac{x(\cancel{1+u^2})\,dx}{x^2\,(\cancel{1+u^2})} = \frac{-\cancel{x^2}(1+u)\,du}{\cancel{x^2}\,(1+u^2)}$$

$$\boxed{\int \frac{1}{1+x^2}dx \quad \tan^{-1}x}$$

$$\frac{1}{x}dx = -\frac{(1+u)}{(1+u^2)}du$$

$$\int \frac{1}{x}dx = \int\left[-\frac{1}{1+u^2} - \frac{u}{1+u^2}\right]du$$

$$\ln x = -\int \frac{1}{1+u^2}du - \frac{1}{2}\int \frac{2u}{1+u^2}du$$

$$\ln x = -\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + c$$

$$\boxed{\int \frac{f'}{f} = \ln f}$$

$$\ln x^2 = -2\tan^{-1}u - \ln(1+u^2) + 2c$$
$$\ln x^2 + \ln(1+u^2) = -2\tan^{-1}u + 2c$$
$$\ln(x^2(1+u^2)) = -2\tan^{-1}u + 2c$$

$$\ln(x^2 + x^2u^2) = -2\tan^{-1}u + 2C$$

$$\ln\left(x^2 + \cancel{x^2}\frac{y^2}{\cancel{x^2}}\right) = -2\tan^{-1}\frac{y}{x} + 2C$$

$$\ln(x^2+y^2) = -2\tan^{-1}\frac{y}{x} + 2C$$

$e \qquad\qquad e$

$$\boxed{x^2 + y^2 = C_1 e^{-2\tan^{-1} y/x}}$$

2.4 Exact

Definition 2.4.1 A differential expression $M(x,y)dx + N(x,y)dy$ is an exact differential in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$ defined in $\mathbb{R}$. A first-order differential equation of the form

$$\underline{M(x,y)}dx + \underline{N(x,y)}dy$$

s said to be an exact equation if the expression on the left-hand side is an exact differential.

Theorem 2.4.1 Let $M(x,y)$ and $N(x,u)$ be continuous and have continuous first partial derivatives in a rectangular region $\mathbb{R}$ defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the solution of differential equation is

$$\int_{(y \text{ constant})} M dx + \int (\text{term of } M \text{ not containing } x) dy = c$$

Exact كيف نتعرف على المعادلة

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{Exact}}$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{Non exact}}$$

Is this equation exact or non exact

$$\underbrace{2xy}_{M}\,dx + \underbrace{x^2}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2x$$

$$\frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

exact

خطوات حل المعادلة exact

① التأكد من ان المعادله هي exact

② $\int M\,dx + \int N\,dy = C$ مهم جداً

نجري التكامل للحد الاول بالنسبة x و الحد الثاني بالنسبة y
ونستثني الحدود المحتوى على x

③ نحسب التكامل و نرتب المعادلة:

■ **Example 2.11** Solve the differential equation $(2x-1)\,dx+(3y+7)\,dy=0$

① نتأكد ان معادله exact

$M = 2x-1 \qquad \frac{\partial M}{\partial y} = 0$

$N = 3y-7 \qquad \frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ it is exact

② $\int (2x-1)\,dx + \int (3y+7)\,dy = C$

$\cancel{2}\frac{x^2}{\cancel{2}} - x + \frac{3y^2}{2} + 7y = c$

$x^2 - x + \frac{3}{2}y^2 + 7y = C$

$e^x = e^x$
$e^y = e^y$

■ **Example 2.12** Solve $(\underbrace{3x^2y+e^y}_{M})\,dx+(\underbrace{x^3+xe^y-2y}_{N})\,dy=0$

$$M = 3x^2y + e^y \qquad \frac{\partial M}{\partial y} = 3x^2 + e^y$$

$$N = x^3 + xe^y - 2y \qquad \frac{\partial N}{\partial x} = 3x^2 + e^y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \checkmark \text{ exact}$$

نتكامل الكل اتش في نقوم بتكامل فقط الحدود التي لا تحتوي على x

$$\int (3x^2y + e^y)\,dx + \int (\cancel{x^3} + \cancel{xe^y} - 2y)\,dy = C$$

$$3y\frac{x^3}{3} + e^y x + \int -2y\,dy$$

$$3y\frac{x^3}{3} + e^y x - \frac{2y^2}{2} = C$$

$$yx^3 + e^y x - y^2 = C$$

$$\overset{M}{}\qquad\overset{N}{}$$

■ **Example 2.13** Solve $\underline{(\sin y - y\sin x)}\,dx + \underline{(\cos x + x\cos y - y)}\,dy = 0$

$M = \sin y - y\sin x \qquad \dfrac{\partial M}{\partial y} = \cos y - \sin x$

$N = \cos x + x\cos y - y \qquad \dfrac{\partial N}{\partial x} = -\sin x + \cos y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{exact}$$

$$\cos y - \sin x = \cos y - \sin x$$

$$\int \sin y - y\sin x \, dx + \int -y \, dy = C$$

$$x\sin y + y\cos x - \frac{y^2}{2} = C$$

■ **Example 2.14** Solve $\underbrace{(4y+2x-5)}_{M}\,dx + (\underbrace{6y+4x-1}_{N})\,dy = 0$

$M = 4y + 2x - 5 \qquad \dfrac{\partial M}{\partial y} = 4$

$N = 6y + 4x - 1 \qquad \dfrac{\partial N}{\partial x} = 4$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

exact

$$\int (4y + 2x - 5)\,dx + \int (6y - 1)\,dy = C$$

$$4yx + \frac{\cancel{2}x^2}{\cancel{2}} - 5x + \frac{\overset{3}{\cancel{6}}y^2}{\cancel{2}} - y = C$$

$$4yx + x^2 - 5x + 3y^2 - y = C$$

■ **Example 2.15** Solve $\underbrace{(x+y)^2}_{M}\,dx+\underbrace{(2xy+x^2-1)}_{N}\,dy=0$

$M=(x+y)^2 \qquad \dfrac{\partial M}{\partial y}=2(x+y)=2x+2y$

$N=2xy+x^2-1 \qquad \dfrac{\partial N}{\partial x}=2y+2x$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \qquad \text{exact}$$

$$\int(x+y)^2dx+\int(\cancel{2xy}+\cancel{x^2}-1)\,dy=c$$

$$\int(x+y)^2dx+\int -dy=c$$

فك التربيع

$$\int(x^2+2xy+y^2)dx+\int -dy=c$$

$$\frac{x^3}{3}+\cancel{2}y\frac{x^2}{\cancel{2}}+y^2x-y=c$$

$$\frac{x^3}{3}+yx^2+y^2x-y=c$$

2.5 Non-exact

There are non-exact differential equations of first-order which can be made into exact differential equations by multiplication with an expression called an integrating factor.

Definition 2.5.1 Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be not an exact differential equation such that

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

If $M(x,y)\,dx + N(x,y)\,dy = 0$ can be made exact multiplying it with a suitable function $\mu(x,y) \neq 0$ then $\mu(x,y)$ is called an integrating factor of $M(x,y)\,dx + N(x,y)\,dy = 0$.

Finding an integrating factor for a non-exact equation discuss as follows:

1. If there exists a continuous single variable function $f(x)$ and $g(x)$ such that

$$\text{If } \frac{M_y - N_x}{N} = f(x) \quad \text{then} \quad IF = e^{\int f(x)\,dx}$$

$$\text{If } \frac{M_y - N_x}{M} = g(y) \quad \text{then} \quad IF = e^{-\int g(y)\,dy}$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{exact}$$

في حال كانت النتيجه

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ Non exact تسمى معادله

$\boxed{\text{خطوات حل معادله Non exac}}$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

1- نتأكد ان المعادله هي Nonexact

2- نحسب معامل التكامل Integral factor

$$\frac{M_y - N_x}{N} = f(x) \qquad IF = e^{\int f(x)\,dx}$$

$$\frac{M_y - N_x}{M} = g(y) \qquad IF = e^{-\int g(y)\,dy}$$

3- نضرب جميع المعادلة ب IF ونتأكد انها اصبحت Exact

4- نحل المعادله بطريقة Exact كما في الدرس السابقه

■ **Example 2.16** Solve $\underset{M}{xy}\,dx + (\underset{N}{2x^2 + 3y^2 - 20})\,dy = 0$

① نتأكد ان المعادله Nonexact

$M = xy$ $N = 2x^2 + 3y^2 - 20$

$\frac{\partial M}{\partial y} = x$ $\frac{\partial N}{\partial x} = 4x$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Nonexact

② $\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} \neq f(x)$

لم ينطبق القانون الاول ننتقل للثاني :

$g(y)$

$\frac{M_y - N_x}{M} = \frac{x - 4x}{xy} = \frac{-3\cancel{x}}{\cancel{x}y} = \frac{-3}{y}$

$IF = e^{-\int -\frac{3}{y}\,dy} = e^{3\int \frac{1}{y}\,dy} = e^{\ln y^3} = y^3$

③ نضرب جميع المعادله ب IF

$y^3 \left[xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0 \right]$

$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$

④ نتأكد ان المعادله اصبحت exact

$\frac{\partial M}{\partial y} = 4xy^3$ $\frac{\partial N}{\partial x} = 4xy^3$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{exact}$$

(٢) نكامل المعادلة

$$\int x y^4 \, dx + \int (\cancel{2x^2y^3} + 3y^5 - 20y^3)\, dy$$

$$\frac{y^4 x^2}{2} + \frac{3y^6}{6} - \frac{20y^4}{4} = C$$

$$\frac{y^4 x^2}{2} + \frac{y^6}{2} - 5y^4 = C$$

■ **Example 2.17** Solve $\underbrace{y(x+y+1)}_{M}\, dx + \underbrace{(x+2y)}_{N}\, dy = 0$

① $(yx + y^2 + y)\, dx + (x+2y)\, dy = 0$

$$\frac{\partial M}{\partial y} = x + 2y + 1 \qquad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

② $$\frac{M_y - N_x}{N} = \frac{x + 2y + \cancel{1} - \cancel{1}}{x+2y} = \frac{\cancel{x+2y}}{\cancel{x+2y}} = 1$$

$$IF = e^{\int F(x)\, dx} = e^{\int 1\, dx} = e^x$$

نضرب جميع المعادلة ب e^x

$$e^{x}(yx+y^2+y)\,dx + e^{x}(x+2y)\,dy = 0$$

$$\Big(\underbrace{yxe^{x}+y^2e^{x}+ye^{x}}_{M}\Big)dx + \Big(\underbrace{xe^{x}+2ye^{x}}_{N}\Big)dy$$

$$\frac{\partial M}{\partial y} = xe^{x} + 2ye^{x} + e^{x} \qquad \frac{\partial N}{\partial x} = xe^{x} + e^{x} + 2ye^{x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int (yxe^{x} + y^2e^{x} + ye^{x})\,dx + \cancel{\int (xe^{x} + 2ye^{x})\,dy} = C$$

$$y\int xe^{x}\,dx + y^2\int e^{x} + y\int e^{x}$$

$$\int xe^{x} = uv - \int v\,du$$

$u = x$, $dv = e^{x}$

$du = 1$

$v = \int e^{x} = e^{x}$

$$= xe^{x} - \int e^{x}$$

$$\boxed{xe^{x} - e^{x}}$$

$$y(xe^{x} - e^{x}) + y^2e^{x} + ye^{x} = C$$

$$xye^{x} - \cancel{ye^{x}} + y^2e^{x} + \cancel{ye^{x}} = C$$

$$xye^{x} + y^2e^{x} = C$$

■ **Example 2.18** Solve $(\underbrace{2y^2+3x}_{M})\,dx+\underbrace{2xy}_{N}\,dy=0$

$$\frac{\partial M}{\partial y}=4y \qquad \frac{\partial N}{\partial x}=2y$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \qquad \text{Nonexact}$$

$$\frac{M_y-N_x}{N}=\frac{4y-2y}{2xy}=\frac{2y}{2xy}=\frac{1}{x}=f(x)$$

$$IF=e^{\int\frac{1}{x}dx}=\cancel{e^{\ln x}}=x$$

نوزع جميع الحدود ⇐ x

$$x\left[(2y^2+3x)\,dx+2xy\,dy\right]$$

$$(2y^2x+3x^2)\,dx+2x^2y\,dy$$

$$\frac{\partial M}{\partial y}=4yx \qquad \frac{\partial N}{\partial x}=4xy$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \qquad \text{exact}$$

$$\int(2y^2x+3x^2)\,dx+\cancel{\int 2x^2y\,dy}=c$$

$$\cancel{2}y^2\frac{x^2}{\cancel{2}}+\cancel{3}\frac{x^3}{\cancel{3}}=c$$

$$x^2y^2+x^3=c$$

2.6 Linear Differential Equations معادلات تفاضليه خطيه

Definition 2.6.1 A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a linear equation in the dependent variable y.

$$\frac{a(x)\frac{dy}{dx} + q(x)y}{} = \frac{g(x)}{}$$

$$3\frac{dy}{dx} + 5xy = \sin x$$

Method of solving Linear Differential Equation can be solved as discuss below:

1. The linear equation of standard form is $\frac{dy}{dx} + p(x)y = q(x)$
2. Integral Factor $(IF) = e^{\int p(x)\,dx}$
3. The solution of differential equation is $y \cdot (IF) = \int q(x) \cdot (IF)\,dx$

الشكل العام للمعادله الخطيه من الدرجه الاولى

$$\boxed{\frac{dy}{dx} + P(x)\,y = q(x)}$$

*خطوات حل المعادلة

① نتأكد ان المعادله خطيه ونحدد قيمه P و q

② نحسب Integral factor

$$IF = e^{\int P(x)\,dx}$$

③ نطبق العلاقة التاليه

$$y\,(IF) = \int q(x)\,(IF)\,dx$$

④ نحل المعادله ونحسب قيمه y

■ **Example 2.19** Solve $\frac{dy}{dx} = 8y$

$$\frac{dy}{dx} = 8y$$

① نتأكد انها خطية

$$\frac{dy}{dx} - 8y = 0$$

$$P(x) = -8 \qquad q(x) = 0$$

② نحسب IF

$$IF = e^{\int P(x)\,dx}$$

$$IF = e^{\int -8\,dx} = e^{-8x}$$

③

$$(IF)y = \int IF\; q(x)\,dx$$

$$y\,e^{-8x} = \int 0\,dx$$

$$y\,e^{-8x} = C$$

$$y = \frac{C}{e^{-8x}}$$

$$y = C\,e^{+8x}.$$

■ **Example 2.20** Solve $\frac{dy}{dx} - 3y = 6$

$$\frac{dy}{dx} - 3y = 6 \qquad p(x) = -3 \qquad q(x) = 6$$

$$IF = e^{\int p(x)\,dx} = e^{\int -3\,dx} = e^{-3x}$$

نطبق القاعدة

$$y\,(IF) = \int (IF)\,q(x)\,dx$$

$$y e^{-3x} = 6 \int e^{-3x}\,dx$$

$$y e^{-3x} = 6\,\frac{e^{-3x}}{-3} + c$$

$$y e^{-3x} = -2 e^{-3x} + c$$

$$y = \frac{-2\cancel{e^{-3x}}}{\cancel{e^{-3x}}} + \frac{c}{e^{-3x}}$$

$$y = -2 + c e^{+3x}$$

■ **Example 2.21** Solve $\frac{dy}{dx}+\frac{2}{x}y=\frac{2}{x}$

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{2}{x} \qquad P(x)=\frac{2}{x} \qquad q(x)=\frac{2}{x}.$$

$$IF = e^{\int P(x)\,dx} = e^{\int \frac{2}{x}dx} = \cancel{e}^{\cancel{\ln} x^2} = x^2$$

$$y\,(IF) = \int (IF)\, q(x)\, dx$$

$$y\,x^2 = \int x^{\cancel{2}}\, \frac{2}{\cancel{x}}\, dx$$

$$y\,x^2 = 2\int x\, dx$$

$$y\,x^2 = \cancel{2}\frac{x^2}{\cancel{2}} + c$$

$$y = \frac{x^2}{x^2} + \frac{c}{x^2}$$

$$y = 1 + cx^{-2}$$

■ **Example 2.22** Solve $x\dfrac{dy}{dx} - 4y = x^6 e^x$

$$\frac{\cancel{x}\,\frac{dy}{dx}}{\cancel{x}} - \frac{4y}{\cancel{x}} = \frac{x^6 e^x}{\cancel{x}}$$

$$\frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \quad \begin{cases} \to P(x) = -\frac{4}{x} \\ \to x^5 e^x \end{cases}$$

$$IF = e^{\int P(x)\,dx} = e^{\int -\frac{4}{x}dx} = e^{\ln x^{-4}} = x^{-4}$$

$$y(IF) = \int IF\, q(x)$$

$$y\,x^{-4} = \int x^{-4} x^5 e^x \, dx$$

$$y\,x^{-4} = \int x e^x \, dx \quad \to \int x e^x = x e^x - e^x$$

تكامل بالاجزاء

$$\frac{y}{x^4} = x e^x - e^x + c$$

$\int x e^x \, dx$

$u = x \qquad du = 1$

$dv = e^x \qquad \int e^x = e^x$

$uv - \int v\,du = x e^x - \int e^x dx$

$$\boxed{= x e^x - e^x}$$

نضرب جميع الحدود ب x^4

$$\boxed{y = x^5 e^x - x^4 e^x + c x^4}$$

■ **Example 2.23** Solve $x\frac{dy}{dx}+(3x+1)y=e^{-3x}$

$$\frac{dy}{dx}+\left(\frac{3x+1}{x}\right)y=\frac{e^{-3x}}{x}$$

$$P(x)=3+\frac{1}{x} \qquad q(x)=\frac{1}{x}e^{-3x}$$

$$IF=e^{\int P(x)\,d}=e^{\int 3+\frac{1}{x}}=e^{3x+\ln x}$$

$$IF=e^{3x}\cancel{e}^{\cancel{\ln}x}=xe^{3x}.$$

$$y(IF)=\int (IF)q(x)$$

$$yxe^{3x}=\int \cancel{x}e^{3x}\frac{1}{\cancel{x}}e^{-3x}\,dx$$

$$yxe^{3x}=\int dx$$

$$yxe^{3x}=x+c$$

$$y=\frac{\cancel{x}}{\cancel{x}e^{3x}}+\frac{c}{xe^{3x}}$$

$$y=e^{-3x}+\frac{c}{x}e^{-3x}$$

■ **Example 2.24** Solve $\frac{dy}{dx} + 2xy = x$

$P(x) = 2x \qquad q(x) = x$

$$IF = e^{\int 2x\,dx} = e^{\frac{\cancel{2}x^2}{\cancel{2}}} = e^{x^2}$$

$$(IF)y = \int (IF)\, q(x)\, dx$$

$$ye^{x^2} = \int x e^{x^2} dx$$

$x^2 = u$

$2x\,dx = du$

$dx = \frac{du}{2x}$

$$ye^{x^2} = \int \cancel{x} e^{u} \frac{du}{2\cancel{x}}$$

$$ye^{x^2} = \frac{1}{2}\int e^{u} du$$

$$ye^{x^2} = \frac{1}{2} e^{x^2} + c$$

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$$y = \frac{1}{2}\frac{\cancel{e^{x^2}}}{\cancel{e^{x^2}}} + \frac{c}{e^{x^2}}$$

$$y = ce^{-x^2} + \frac{1}{2}$$

■ **Example 2.25** Solve $(x^2-9)\frac{dy}{dx}+xy=0$

$$\frac{(x^2-9)}{x^2-9}\frac{dy}{dx} + \frac{xy}{(x^2-9)} = \frac{0}{(x^2-9)}$$

$$\frac{dy}{dx} + \frac{x}{x^2-9}\,y = 0$$

$$P(x) = \frac{x}{x^2-9} \qquad q(x) = 0$$

$$IF = e^{\int \frac{x}{x^2-9}} = e^{\ln(x^2-9)^{1/2}}$$

$$\frac{1}{2}\int \frac{2x}{x^2-9} = \frac{1}{2}\ln x^2-9$$

$$IF = (x^2-9)^{1/2}$$

$$y(IF) = \int (IF)\, q(x)\, dx$$

$$y\sqrt{x^2-9} = \int 0$$

$$y\sqrt{x^2-9} = c$$

$$y = \frac{c}{\sqrt{x^2-9}}$$

2.7 Bernolli's

An equation of the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$, where P and Q are functions of x only or constants, n is real constant is known as Bernoulli's DE (Equations reducible to the linear form). Though not linear, it can be made linear.

Method of solving Bernolli's Equation can be solved as discuss below:

1. Divide with y^n of given equation, we get $y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x)$
2. Let $z=y^{(1-n)} \Longrightarrow (1-n)y^{-n}\frac{dy}{dx}=\frac{dz}{dx} \Longrightarrow y^{-n}\frac{dy}{dx}=\left(\frac{1}{1-n}\right)\frac{dz}{dx}$
3. By substitution in (1), we get $\frac{dz}{dx}+(1-n)p(x)z=(1-n)q(x)$ is linear z.

$\frac{dy}{dx}+P(x)y=q(x)$ المعادله الخطيه

$\frac{dy}{dx}+P(x)y=q(x)y^n$ معادله برنولي

خطوات الحل

① نتأكد ان المعادله هي معادله برنولي ونتحقق من

$P(x)$ $q(x)$ n

② نفرض ان

$z=y^{(1-n)}$

③ نعوض قيم P و q و n في

$$\boxed{\frac{dz}{dx}+(1-n)P(x)z=(1-n)q(x)}$$ حفظ

لنحصل على معادله خطيه

④ نحل المعادلة الخطيه بدلالة z

⑤ نعوض مكان z بدلالة y

■ **Example 2.26** Solve the differential equation $\frac{dy}{dx} + \frac{1}{x}y = 3y^3$

$$\frac{dy}{dx} + \frac{1}{x}y = 3y^3 \quad (1)$$

$$P(x) = \frac{1}{x} \qquad q(x) = 3 \qquad n = 3$$

$$z = y^{1-n} = y^{1-3} = y^{-2} \quad (2)$$

$$\boxed{z = y^{-2}}$$

$$(3)$$

$$\frac{dz}{dx} + (1-n)\,P(x)\,z = (1-n)\,q(x)$$

$$\frac{dz}{dx} + (1-3)\left(\frac{1}{x}\right)z = (1-3)\,3$$

$$\frac{dz}{dx} - \underbrace{\frac{2}{x}}_{P}\, z = \underbrace{-6}_{q} \quad (4)$$

معادلة خطية

$$IF = e^{\int -\frac{2}{x}} = e^{\ln x^{-2}} = \cancel{e^{\ln}}\, x^{-2} = x^{-2}$$

$$(If)\,z = \int (If)(-6)$$

$$x^{-2} z = -6\int x^{-2}\,dx = -6\,\frac{x^{-1}}{-1} + c$$

نضرب ب x^2 $\qquad \frac{z}{x^2} = \frac{6}{x} + c \qquad z = 6x + cx^2$

$$\frac{1}{y^2} = 6x + cx^2 \qquad y^2 = \frac{1}{6x + cx^2}$$

■ **Example 2.27** Solve the differential equation $\frac{dy}{dx} - y = e^x y^2$

$$\frac{dy}{dx} - y = e^x y^2$$ ①

$P(x) = -1 \qquad q(x) = e^x \qquad n = 2$

نفرض ②

$$z = y^{1-n} = y^{1-2} = \frac{1}{y}$$

③

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)q(x)$$

$$\frac{dz}{dx} + (1-2)(-1)z = (1-2)e^x$$

$$\frac{dz}{dx} + z = -e^x$$ معادلة خطية

④ حل المعادلة الخطية

$P(x) = 1 \qquad q = -e^x$

$$IF = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$$

$$z(IF) = \int (IF)\,q(x)$$

$$ze^x = -\int e^x e^x\,dx = -\int e^{2x} = -\frac{e^{2x}}{2} + c$$

$$\frac{ze^x}{e^x} = \frac{\frac{-e^{2x}}{2}}{e^x} + \frac{c}{e^x} \qquad z = -\frac{e^x}{2} + ce^{-x}$$

⑤

$$\frac{1}{y} = -\frac{e^x}{2} + ce^{-x} \qquad y = \frac{1}{-\frac{e^x}{2} + ce^{-x}}$$

■ **Example 2.28** Solve $x\dfrac{dy}{dx}-(x+1)y=xy^2$

$$x\frac{dy}{dx}-(x+1)y=xy^2$$

نقسم كل المعادلة
على x

$$\frac{dy}{dx}-\left(\frac{x+1}{x}\right)y=y^2 \qquad (1)$$

$$\frac{dy}{dx}-\left(1+\frac{1}{x}\right)y=1y^2$$

$$P(x)=-\left(1+\frac{1}{x}\right) \qquad q(x)=1 \qquad n=2$$

$$z=y^{1-n}=y^{1-2}=y^{-1}=\frac{1}{y} \qquad (2)$$

$$\frac{dz}{dx}+(1-n)P(x)z=(1-n)q(x) \qquad (3)$$

$$\frac{dz}{dx}+-(-1)\left[1+\frac{1}{x}\right]z=(-1)(1)$$

$$\frac{dz}{dx}+\left(1+\frac{1}{x}\right)z=-1$$

$$P(x)=1+\frac{1}{x} \qquad q(x)=-1$$

$$IF=e^{\int P(x)\,dx}=e^{\int 1+\frac{1}{x}}=e^{x+\ln x}$$

$$If = e^{x}\cancel{e^{\ln x}} = xe^{x}$$

$$Z(If) = \int (If)\, q(x)$$

$$Z\, xe^{x} = -\int xe^{x}$$

$xe^{x} - e^{x}$ بالأجزاء

$$Zxe^{x} = -(xe^{x} - e^{x}) + C$$

بنقسم كل المعادله ع xe^{x}

$$\frac{Z\cancel{xe^{x}}}{\cancel{xe^{x}}} = \frac{-\cancel{xe^{x}}}{\cancel{xe^{x}}} - \frac{\cancel{e^{x}}}{x\cancel{e^{x}}} + \frac{C}{xe^{x}}$$

$$Z = -1 - \frac{1}{x} + \frac{c}{x}e^{-x}$$

$$Z = \frac{1}{y}$$

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$$\frac{1}{y} = -\frac{x}{x} - \frac{1}{x} + \frac{c}{x}e^{-x}$$

$$\frac{1}{y} = \frac{-x-1+ce^{-x}}{x}$$

$$y = \frac{x}{-x-1+ce^{-x}}$$