CHAPTER 1

The Wave Function

1.1 The Schrodinger equation

$x(t)$

$v = \frac{dx}{dt}$

$a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$

$F = ma$

$T = \frac{1}{2} m v^2$

$F = -\frac{\partial V}{\partial x}$

$U = Fx$

باستخدام قوانين الميكانيكا الكلاسيكية من السهل التعرف على موقع وحركة وتسارع الجسم

✓ اذا اردنا معرفه موقع الالكترون نتعامل مع الالكترون كأنه موجة

$\Psi(x,t)$

wave function

Schrodinger equation

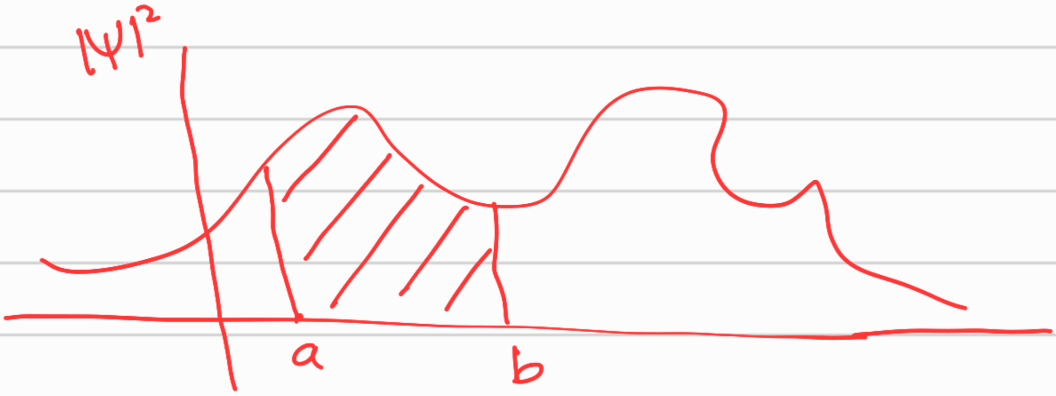
$$\underbrace{i\hbar \frac{\partial \psi}{\partial t}}_{\text{Total energy}} = \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}}_{\text{KE طاقة حركية}} + \underbrace{V\psi}_{\text{طاقة الوضع PE}}$$

$i = \sqrt{-1}$

$$\hbar = \frac{h}{2\pi} = \frac{6.6 \times 10^{-34}}{2\pi}$$

$$\hbar = 1.0545 \times 10^{-34}\ Js$$

1.2 the statistical Interpetation



$$\int_a^b |\psi|^2 \, dx$$

التعبير الاحصائي عن
احتماليه وجود الجسم
في مكان ما في المنطقه من
$a \to b$

$\int_a^b |\psi(x,t)|^2 \, dx$ = the Probability of finding the particle betweon a and b

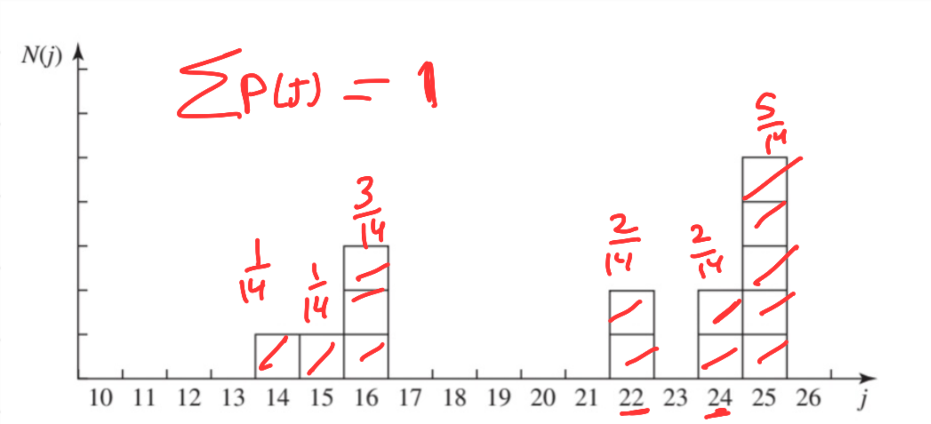
$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \, dx = 1$$

the particle is at point c

1.3 Probability

* Discrete Variables

$N(14) = 1$
$N(15) = 1$
$N(16) = 3$
$N(22) = 2$
$N(24) = 2$
$N(25) = 5$



Question 1 If you selected one individual at random from this group, what is the **probability** that this person's age would be 15?

$$P(15) = \frac{N(15)}{N}$$

$$= \frac{1}{14}$$

$$P(j) = \frac{N(j)}{N}$$

$P(j)$ احتمالية عمر معين

N العدد الكلي

$N(j)$ عدد أفراد عمر معين

Question 2 What is the **most probable** age?

25

Question 3 What is the **median** age?

الفئه الي تأتي
بالوسط

23

Question 4 What is the **average** (or **mean**) age?

$$\boxed{\langle j \rangle = \frac{j\,N(j)}{N} = \sum j\,P(j)}$$

$$\langle j \rangle = \frac{14(1) + 15(1) + 16(3) + 22(2) + 24(2) + 25(5)}{14}$$

$$\langle j \rangle = 21$$

$$14 \times \frac{1}{14} + 15 \times \frac{1}{14} + 16 \times \frac{3}{14} + 22 \times \frac{2}{14} + 24 \times \frac{2}{14} + 25 \times \frac{5}{14}$$

j $P(j)$

$$= 21$$

$$\langle j \rangle^2 = 21^2 = 441$$

Question 5 What is the average of the *squares* of the ages?

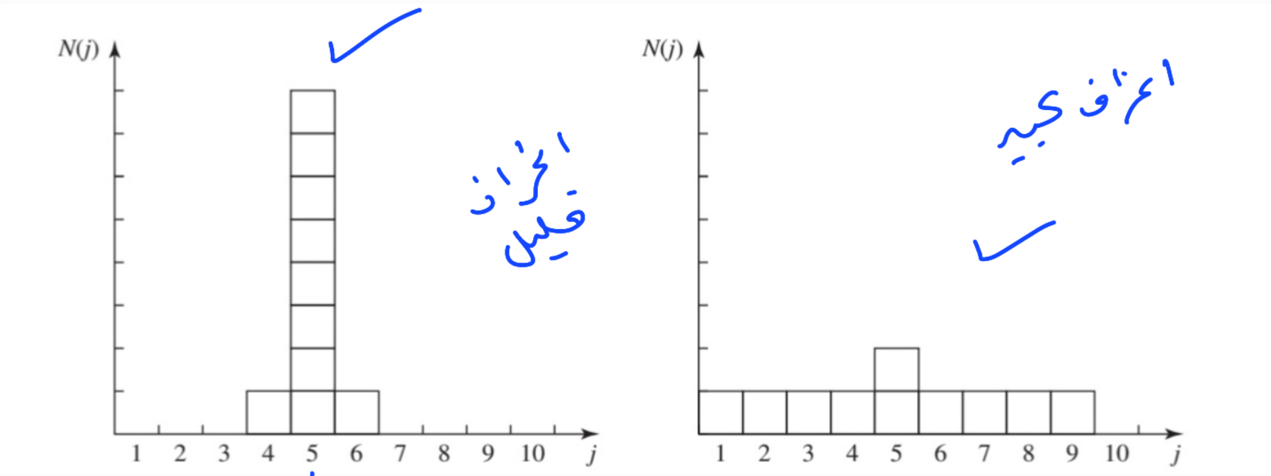
$$\langle j^2 \rangle = \sum j^2\,P(j)$$

$$\langle f(j) \rangle = \sum f(j)\,P(j)$$

$$= 14^2 \times \frac{1}{14} + 15^2 \times \frac{1}{14} + 16^2 \times \frac{3}{14} + 22^2 \times \frac{2}{14} + 24^2 \times \frac{2}{14} + 25^2 \times \frac{5}{14}$$

$$\langle j^2 \rangle = 459.6$$

$$\langle j \rangle^2 \neq \langle j^2 \rangle$$



mean
median = 5

mean
median = 5

standard deviation الانحراف المعياري

σ^2

تباين $\quad \sigma \rightarrow$ variance

$$\boxed{\begin{aligned} \sigma^2 &= \langle j^2 \rangle - \langle j \rangle^2 \\ \sigma &= \sqrt{\langle j^2 \rangle - \langle j \rangle^2} \end{aligned}}$$

1.3.2 Continuous variables

Probability $= \rho(x)\, \underline{dx}$

احتماليه اختيار عدد في فتره معينه dx

Probability density $a \longrightarrow b$

$$\boxed{P_{ab} = \int_a^b \rho(x)\, dx}$$

$$\int_{-\infty}^{\infty} \rho(x)\, dx = 1$$

$$\boxed{\langle f(x) \rangle = \int_{-\infty}^{+\infty} f(x)\, \rho(x)\, dx}$$

$$\boxed{\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2}$$

حل تمارين كم (1) شابتر (1)

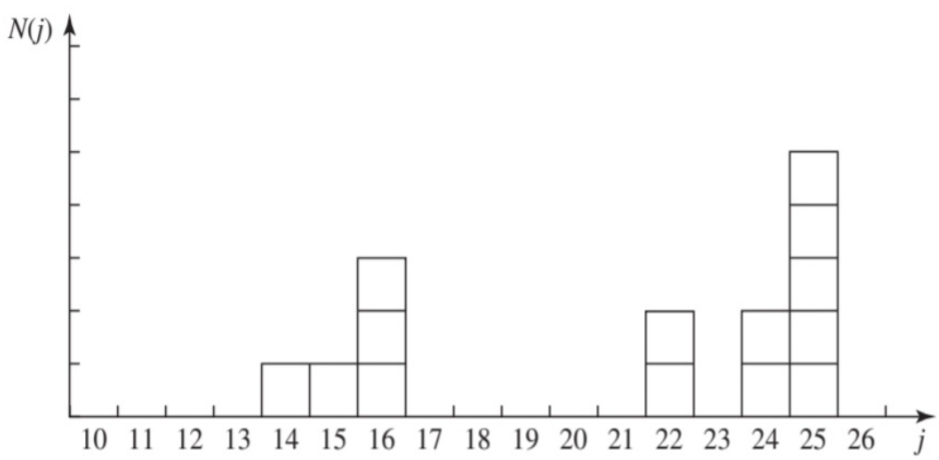
1.1 + 1.2 + 1.3

Problem 1.1 For the distribution of ages in the example in Section 1.3.1:

(a) Compute $\langle j^2 \rangle$ and $\langle j \rangle^2$.

(b) Determine Δj for each j, and use Equation 1.11 to compute the standard deviation.

(c) Use your results in (a) and (b) to check Equation 1.12.



J	N(J)
14	1
15	1
16	3
22	2
24	2
25	5
	14

a) $\langle J^2 \rangle$ $\quad \langle J \rangle^2$

$$\langle J \rangle = \sum J P(J) = \sum \frac{J\, N(J)}{N}$$

$$\langle J \rangle = \frac{14(1) + 15(1) + 16(3) + 22(2) + 24(2) + 25(5)}{14}$$

$$= 21$$

$$\langle J \rangle^2 = 21^2 = 441$$

$$\langle J^2 \rangle = \sum J^2 P(J) = \sum J^2 \frac{N(J)}{N}$$

$$\langle J^2 \rangle = \frac{14^2(1) + 15^2(1) + 16^2(3) + 22^2(2) + 24^2(2) + 25^2(5)}{14}$$

$$\langle J^2 \rangle = 459.57$$

b)

$$\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2}$$

(1.11) $$\sigma^2 = \langle \Delta j^2 \rangle = \frac{\sum (\Delta j)^2 N(j)}{N}$$

j	$\Delta j = j - \langle j\rangle$	$N(j)$
14	$14-21 = -7$	1
15	$15-21 = -6$	1
16	$16-21 = -5$	3
22	$22-21 = 1$	2
24	$24-21 = 3$	2
25	$25-21 = 4$	5

$$\sigma^2 = \frac{\sum \Delta j^2 \, N(j)}{N}$$

$$= \frac{(-7)^2 1 + (-6^2)1 + (-5)^2(3) + 1(2) + 3^2(2) + 4^2(5)}{14}$$

$\sigma^2 = 18.571$ $\qquad$ $\sigma = 4.309$

c)

$$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2$$

$$= 459.571 - 441$$

$$\sigma^2 = 18.571$$

$$\sigma = 4.309$$

Problem 1.2

σ^2

(a) Find the standard deviation of the distribution in Example 1.2.

(b) What is the probability that a photograph, selected at random, would show a distance x more than one standard deviation away from the average?

$$P(x) = \frac{1}{2\sqrt{hx}} \qquad (0 \leq x \leq h)$$

a)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot P(x)\, dx$$

$$\langle x \rangle = \int_0^h x \frac{1}{2\sqrt{hx}} = \int_0^h \frac{x^1}{\sqrt{x}} \frac{1}{2\sqrt{h}}$$

$$\langle x \rangle = \int_0^h \frac{x^{1/2}}{2\sqrt{h}}\, dx = \frac{1}{2\sqrt{h}} \int_0^h x^{1/2}\, dx$$

$$\langle x \rangle = \frac{1}{2\sqrt{h}} \left[\frac{x^{3/2}}{3/2} \right]_0^h$$

$$\langle x \rangle = \frac{1}{\cancel{2}\sqrt{h}} \frac{\cancel{2}}{3} x^{3/2} \Big]_0^h$$

$$\langle x \rangle = \frac{1}{3\sqrt{h}} \left[h^{3/2} - 0^{3/2} \right]$$

$$\langle x \rangle = \frac{1}{3} \frac{h^{3/2}}{h^{1/2}} = \frac{h}{3}$$

$$\langle x \rangle = \frac{h}{3}$$

$$\langle x \rangle^2 = \frac{h^2}{9}$$

$$\langle x^2 \rangle = \int_0^h x^2 \cdot \rho(x)$$

$$= \int_0^h x^2 \frac{1}{2\sqrt{hx}}$$

$$= \frac{1}{2\sqrt{h}} \int_0^h \frac{x^2}{x^{1/2}} = \frac{1}{2\sqrt{h}} \int_0^h x^{3/2}\,dx$$

$$= \frac{1}{2\sqrt{h}} \left. \frac{x^{5/2}}{5/2} \right]_0^h = \frac{1}{\cancel{2}\sqrt{h}} \frac{\cancel{2}}{5} \left[h^{5/2} - 0 \right]$$

$$= \frac{1}{5\,h^{1/2}} h^{5/2} = \frac{h^2}{5}$$

$$\langle x^2 \rangle = \frac{h^2}{5}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{h^2}{5} - \frac{h^2}{9}$$

✓ $$\sigma^2 = h^2\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{4h^2}{45}$$

✓ $$\sigma = \sqrt{\frac{4}{45}}\, h$$

Problem 1.3 Consider the **gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are positive real constants. (The necessary integrals are inside the back cover.)

(a) Use Equation 1.16 to determine A.

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

(c) Sketch the graph of $\rho(x)$.

$$\int_{-\infty}^{+\infty} P(x)\, dx = 1$$

$$\rho(x) = A e^{-\lambda(x-a)^2}$$

$$1 = \int_{-\infty}^{\infty} P(x)\, dx = A \int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\, dx$$

بالتعويض مكان

$$\rightarrow u = x - a$$

$$\rightarrow du = dx$$

$$1 = A \int_{-\infty}^{\infty} e^{-\lambda u^2}\, du$$

$$\int_{0}^{\infty} e^{-au^2}\, du = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$1 = 2A \int_{0}^{\infty} e^{-\lambda u^2}\, du$$

$$1 = \cancel{2}A \left[\frac{1}{\cancel{2}} \sqrt{\frac{\pi}{\lambda}} \right]$$

$$1 = A \frac{\sqrt{\pi}}{\sqrt{\lambda}} \qquad A = \frac{\sqrt{\lambda}}{\sqrt{\pi}} = \sqrt{\frac{\lambda}{\pi}}$$

b) $\langle x \rangle = \int_{-\infty}^{+\infty} x \, P(x) \, dx$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \sqrt{\frac{\lambda}{\pi}} \, e^{-\lambda (x-a)^2} \, dx$$

$u = x - a \qquad x = u + a \qquad$ بتعويض

$du = dx$

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{+\infty} x \, e^{-\lambda u^2} \, du$$

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (u+a) \, e^{-\lambda u^2} \, du$$

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \left[\underbrace{\int_{-\infty}^{\infty} u \, e^{-\lambda u^2} \, du}_{\text{odd function}} + a \int_{-\infty}^{\infty} e^{-\lambda u^2} \, du \right]$$

odd function

= zero

$$\langle x \rangle = \sqrt{\frac{\lambda}{\pi}} \left[0 + 2a \int_{0}^{\infty} e^{-\lambda u^2} \, du \right]$$

$$\langle x \rangle = \cancel{\sqrt{\frac{\lambda}{\pi}}} \left[\cancel{2} a \, \frac{1}{\cancel{2}} \cancel{\sqrt{\frac{\pi}{\lambda}}} \right]$$

$$\langle x \rangle = a$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 \, p(x) \, dx$$

$$= \int x^2 \sqrt{\frac{\lambda}{\pi}} \, e^{-\lambda (x-a)^2} \, dx$$

$u = x - a$

$du = dx$

$x = u + a$

$(u+a)^2 = \underline{u^2} + \underline{2au} + \underline{a^2}$

$$= \sqrt{\frac{\lambda}{\pi}} \int (u+a)^2 e^{-\lambda u^2} \, du$$

فك التربيع

$$= \sqrt{\frac{\lambda}{\pi}} \left[\int_{-\infty}^{+\infty} u^2 e^{-\lambda u^2} + \underbrace{2a \int u e^{-\lambda u^2}}_{\text{zero}} + \underbrace{a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2} du}_{} \right]$$

$$= \sqrt{\frac{\lambda}{\pi}} \left[2 \int_0^{\infty} u^2 e^{-\lambda u^2} \, du + 2a^2 \int_0^{\infty} e^{-\lambda u^2} \, du \right]$$

$$= \cancel{\sqrt{\frac{\lambda}{\pi}}} \left[\cancel{2} \, \frac{1}{\cancel{4}\lambda} \cancel{\sqrt{\frac{\pi}{\lambda}}} + \cancel{2} a^2 \frac{1}{\cancel{2}} \cancel{\sqrt{\frac{\pi}{\lambda}}} \right]$$

2

$$\langle x^2 \rangle = \frac{1}{2\lambda} + a^2$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma = \sqrt{\frac{1}{2\lambda} + \cancel{a^2} - \cancel{a^2}}$$

$$\sigma = \sqrt{\frac{1}{2\lambda}}$$

1.4 Normalization

ψ wave function

1- ψ يجب ان تحقق معادلة شرودنغر

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial t^2} - \frac{i}{\hbar} V\psi$$

ψ ψ^*

i $-i$

$$\frac{\partial \psi^*}{\partial t} = \frac{-i\hbar}{2m}\frac{\partial^2 \psi}{\partial t^2} + \frac{i}{\hbar} V\psi$$

2- $\int |\psi^2|\, dx$ probability density

$$\int_{-\infty}^{\infty} |\psi^2|\, dx = 1$$

Normalized $\int_{-\infty}^{\infty} |\psi^2|\, dx = \int_{-\infty}^{\infty} \psi\psi^*\, dx = 1$

ψ non-normalized $\int_{-\infty}^{\infty} |\psi^{*}| \neq 1$

$$\int_{-\infty}^{\infty} \underline{A}\psi = 1$$

Normalization : نحسب الثابت A الذي يجعل $\int_{-\infty}^{\infty} \psi\psi^* =$

$$\int_{-\infty}^{\infty} |\psi|^2 = 1 \qquad \frac{d}{dt}\int_{-\infty}^{\infty} |\psi|^2 = 0$$

Problem 1.4 At time $t = 0$ a particle is represented by the wave function

$$\Psi(x, 0) = \begin{cases} A(x/a), & 0 \le x \le a, \\ A(b-x)/(b-a), & a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are (positive) constants.

(a) Normalize Ψ (that is, find A, in terms of a and b).

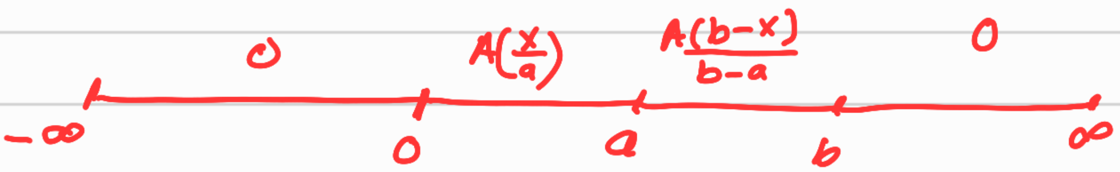
(b) Sketch $\Psi(x, 0)$, as a function of x.

(c) Where is the particle most likely to be found, at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

(e) What is the expectation value of x?

القيمة المتوقعة $\quad \langle x \rangle = \int x \psi^2 dx$



a) $$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$

$$= \int_{-\infty}^{0} |\psi|^2 dx + \int_{0}^{a} |\psi|^2 dx + \int_{a}^{b} |\psi|^2 dx + \int_{b}^{\infty} |\psi^2| dx = 1$$

$$= \cancel{\int_{-\infty}^{0} 0} + \int_{0}^{a} A^2 \frac{x^2}{a^2} dx + \int_{a}^{b} \frac{A^2 (b-x)^2}{(b-a)^2} dx + \cancel{\int_{b}^{\infty} 0} = 1$$

$$= \frac{A^2}{a^2} \int_{0}^{a} x^2 dx + \frac{A^2}{(b-a)^2} \int (b-x)^2 dx = 1$$

$$1 = \frac{A^2}{a^2}\left[\frac{x^3}{3}\right]_0^a + \frac{A^2}{(b-a)^2}\left[\frac{(b-x)^3}{-3}\right]_a^b$$

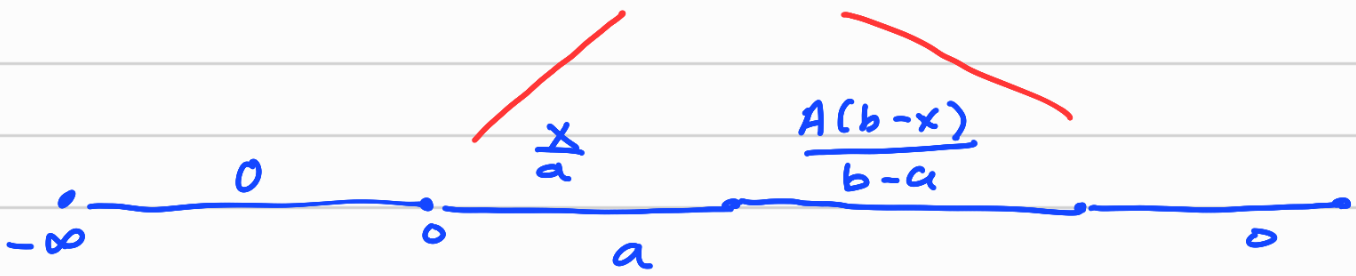
$$1 = \frac{A^2}{a^2}\left[\frac{a^3}{3}\right] + \frac{A^2}{-3(b-a)^2}\left[\cancel{(b-b)^3} - (b-a)^3\right]$$

$$= \frac{A^2}{\cancel{a^2}}\,\frac{\overset{a}{\cancel{a^3}}}{3} + \frac{A^2}{3\cancel{(b-a)^2}}\,\overset{(b-a)}{\cancel{(b-a)^3}} = 1$$

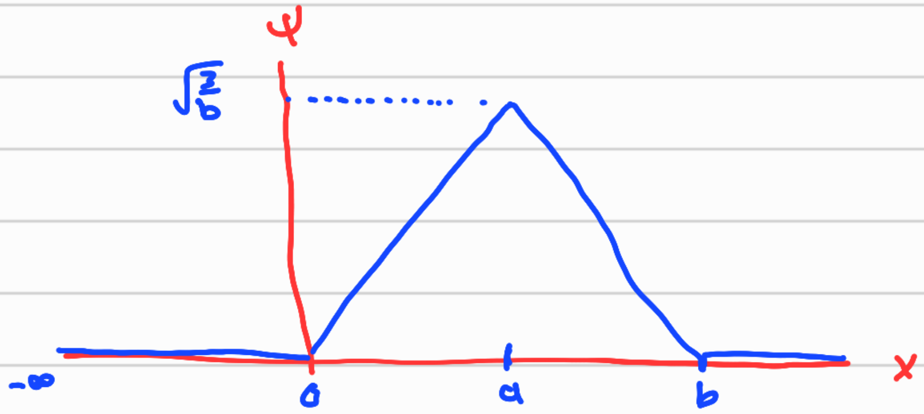
$$1 = \frac{A^2 \cancel{a}}{\cancel{3}} + \frac{A^2 b}{3} - \frac{A^2 \cancel{a}}{\cancel{3}}$$

$$1 = \frac{A^2 b}{3}$$

$$\frac{3}{b} = A^2 \qquad A = \sqrt{\frac{3}{b}}$$



b)



c) at $x = a$

d) $$P = \int_{-\infty}^{a} |\psi(x,0)|^2 \, dx$$

$$P = \int_{-\infty}^{0} \cancel{0 \, dx} + \int_{0}^{a} A^2 \frac{x^2}{a^2} \, dx$$

$$P = \int_0^a \sqrt{\frac{3}{b}}^{\,2} \frac{x^2}{a^2}$$

$$P = \frac{3}{ba^2} \int_0^a x^2 = \frac{3}{ba^2} \left[\frac{x^3}{3} \right]_0^a$$

$$= \frac{3}{ba^2} \frac{a^3}{3} = \frac{a}{b}$$

$$\boxed{P = \frac{a}{b}}$$

$\Rightarrow$ in case $a = b$ $\quad P = \frac{a}{b} = \frac{a}{a} = 1$

in case $b = 2a$ $\quad P = \frac{a}{b} = \frac{a}{2a} = \frac{1}{2}$

$$\begin{cases} a = b & P = 1 \\ b = 2a & P = 1/2 \end{cases}$$

e)

$$\langle x \rangle = \int_{-\infty}^{\infty} \underline{x}\, |\psi|^2 \, dx \qquad A = \sqrt{\tfrac{3}{b}}$$

$$= \int_0^a x \frac{A^2 x^2}{a^2} dx + \int_a^b \frac{x A^2 (b-x)^2}{(b-a)^2} dx$$

$$= \frac{3}{ba^2} \int_0^a x^3 \, dx + \frac{3}{b(b-a)^2} \int_a^b \underbrace{x}_{u} \underbrace{(b-x)^2}_{dv} dx$$

تكامل بالأجزاء

$$\frac{3}{ba^2} \left. \frac{x^4}{4} \right]_0^a + \frac{3}{b(b-a)^2} \left[\frac{x(b-x)^3}{-3} - \int \frac{(b-x)^3}{-3} \right]$$

$u = x \qquad dv = (b-x)^2$

$du = dx \qquad V = \frac{(b-x)^3}{-3}$

$uv - \int v\,du$

$$\frac{3a^4}{4ba^2} + \frac{3}{b(b-a)^2} \left[\frac{a(b-a)^3}{-3} - \frac{(b-x)^4}{12} \right]_a^b$$

$$\frac{3a^2}{4b} + \frac{3}{b(b-a)^2} \left[\frac{a(b-a)^3}{-3} + \frac{(b-a)^4}{12} \right]$$

$$\frac{3a^2}{4b} - \frac{3a(b-a)^3}{3b(b-a)^2} + \frac{3(b-a)^4}{12b(b-a)^2}$$

$$\frac{3a^2}{4b} + \frac{a(b-a)}{b} + \frac{(b-a)^2}{4b}$$

$$\frac{3\cancel{a^2}}{\cancel{4}b} + \frac{4ab}{4b} - \frac{4\cancel{a^2}}{\cancel{4}b} + \frac{b^2}{4b}\left(-\frac{2ab}{4b} + \frac{\cancel{a^2}}{\cancel{4}b}\right)$$

$$= \frac{2ab}{4b} + \frac{b^2}{4b}$$

1.5 Momentum

$$\vec{p} = mv = m\frac{dx}{dt}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x\,|\psi(x,t)|^2\,dx$$

$$\langle v \rangle = \frac{d\langle x \rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} x\,\psi^2\,dx$$

$$\langle v \rangle = \frac{d\langle x \rangle}{dt} = \frac{-i\hbar}{m}\int \psi^* \frac{\partial \psi}{\partial x}\,dx$$

$$p = m\frac{d\langle x \rangle}{dt} = \cancel{m}\left(\frac{-\hbar}{2\cancel{m}}\int \psi^* \frac{\partial \psi}{\partial x}\,dx\right)$$

$$P = -i\hbar \int \psi^* \frac{\partial \psi}{\partial x}\, dx$$

شكل آخر لمعادلة القيمة المتوقعة للموقع والزخم

$$\langle x \rangle = \int \psi^* [x]\, \psi\, dx$$

operator

$$\langle p \rangle = \int \psi^* \left[-i\hbar \frac{\partial}{\partial x}\right] \psi\, dx$$

operator

operator

$$\langle x \rangle \Longrightarrow [x]$$

$$\langle p \rangle \Longrightarrow \left[-i\hbar \frac{\partial}{\partial x}\right]$$

$$\langle Q \rangle \Longrightarrow [Q]$$

Kinetic energy T

$$T = \frac{1}{2} m v^2 = \frac{p^2}{2m}$$

$$p^2 = m^2 v^2 \qquad v^2 = \frac{p^2}{m^2}$$

$$T = \frac{1}{2} m v^2$$

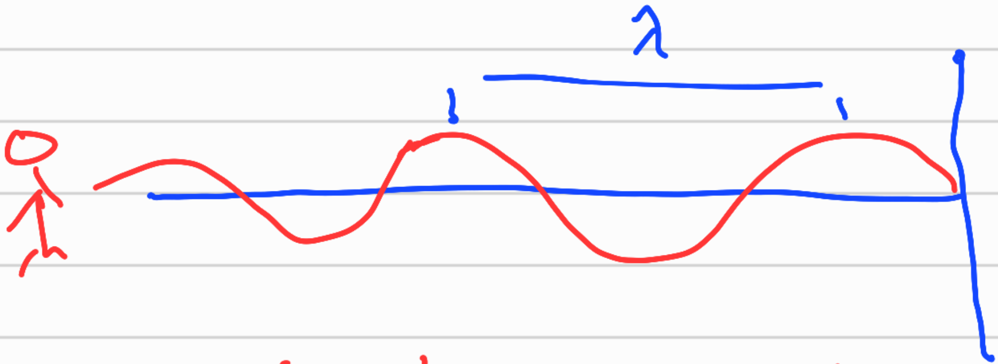
$$T = \frac{1}{2} \cancel{m} \frac{p^2}{m^{\cancel{2}}}$$

$$\langle T \rangle = -\frac{\hbar^2}{2m} \int \psi^* \frac{\partial^2 \psi}{\partial x^2} \, dx$$

$i^2 = -1$

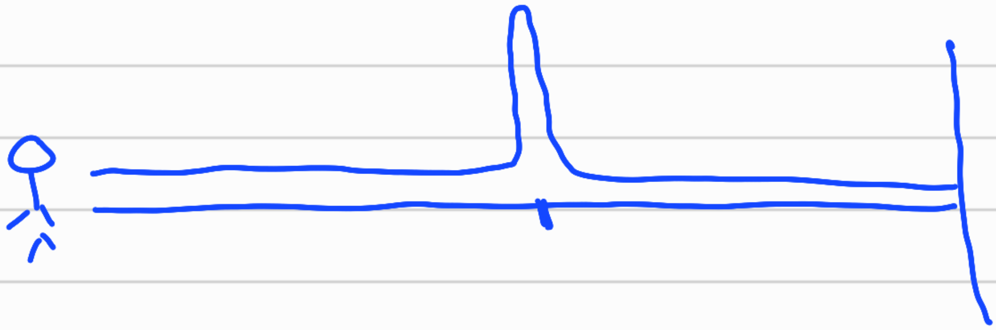
1.6 The uncertainity principle

فرضية الشك عدم الدقة



- we can't determine the exact position of the wave

- yes we can determine the wavelength



✓ We able to define more accurat <u>position</u> of the wave

✓ We can't accurately find the <u>wavelength</u>

اذا ركزنا القياس على موقع الجسم سنحصل
على قياس غير دقيق للزخم والعكس صحيح

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

x : Position $\qquad$ p : momentum

= de Broglie formula

علاقة تربط بين طول الموجه وزخمها

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda}$$

كلما كان طول الموجة أقل كان الزخم أكبر

$1.15 + 1.7 + 1.5$ حل مسائل

Problem 1.5 Consider the wave function

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}, \quad \Psi^* = Ae^{-\lambda|x|}e^{i\omega t}$$

where A, λ, and ω are positive real constants. (We'll see in Chapter 2 for what potential (V) this wave function satisfies the Schrödinger equation.)

(a) Normalize Ψ.

(b) Determine the expectation values of x and x^2.

(c) Find the standard deviation of x. Sketch the graph of $|\Psi|^2$, as a function of x, and mark the points $(\langle x\rangle+\sigma)$ and $(\langle x\rangle-\sigma)$, to illustrate the sense in which σ represents the "spread" in x. What is the probability that the particle would be found outside this range?

a) $$\int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-\infty}^{\infty} \Psi \Psi^* dx = 1$$

$$1 = \int_{-\infty}^{\infty} Ae^{-\lambda|x|} e^{-i\omega t} \cdot Ae^{-\lambda|x|} e^{i\omega t} dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx$$

$$|x| = \begin{cases} -x & x \le 0 \\ x & x \ge 0 \end{cases}$$

$$= A^2 \left(\int_{-\infty}^{0} e^{2\lambda x} dx + \int_{0}^{\infty} e^{-2\lambda x} dx \right)$$

$$\int e^{3x} \quad \frac{e^{3x}}{3}$$

$$1 = A^2 \left(\frac{e^{2\lambda x}}{2\lambda} \Big]_{-\infty}^{0} + \frac{e^{-2\lambda x}}{-2\lambda} \Big]_{0}^{\infty} \right)$$

$$1 = A^2 \left[\frac{1}{2\lambda} - 0 \right] + \left[0 - \frac{1}{-2\lambda} \right]$$

$$e^{-\infty} \rightsquigarrow 0$$
$$e^{\infty} \rightsquigarrow \infty$$

$$1 = A^2 \left[\frac{1}{2\lambda} + \frac{1}{2\lambda} \right]$$

$$1 = A^2 \frac{\cancel{2}}{\cancel{2}\lambda} \qquad 1 = \frac{A^2}{\lambda}$$

$$A^2 = \lambda \qquad A = \sqrt{\lambda}$$

$$\psi(x,t) = \sqrt{\lambda}\, e^{-\lambda|x|} e^{-i\omega t}$$

(b) $$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = \int_{-\infty}^{\infty} x \, \psi \psi^* dx$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{i\omega t} \cdot \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{i\omega t} dx$$

$$\langle x \rangle = \lambda \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx \longrightarrow \overset{\text{odd}}{\int = 0}$$

$$= \text{zero}$$

$$\langle x^2 \rangle = \int x^2 |\psi|^2 dx$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{-i\omega t} \sqrt{\lambda} e^{-\lambda|x|} \underline{e}^{i\omega t} dx$$

$$\langle x^2 \rangle = \lambda \int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx = 2\lambda \int_0^{\infty} x^2 e^{-2\lambda x} dx$$

$$\langle x^2\rangle = 2\lambda\int_0^\infty x^2 e^{-2\lambda x}$$

$u = x^2$ $\quad dv = e^{-2\lambda x}$

$du = 2x$ $\quad v = \frac{e^{-2\lambda x}}{-2\lambda}$

$$\langle x^2\rangle = 2\lambda\left(\left.\frac{x^2 e^{-2\lambda x}}{2\lambda}\right]_0^\infty + \int_0^\infty \frac{x e^{-2\lambda x}}{-\lambda}\right)$$

$uv - \int v\,du$

$[0-0]$

0

$$\langle x^2\rangle = 2\lambda\int_0^\infty \frac{x e^{-2\lambda x}}{\lambda}$$

$$\langle x^2\rangle = \frac{2\cancel{\lambda}}{\lambda}\int_0^\infty x e^{-2\lambda x}\,dx$$

$u = x$ $\quad du = 1$

$dv = e^{-2\lambda x}$ $\quad v = \frac{e^{-2\lambda x}}{-2\lambda}$

$$\langle x^2\rangle = 2\left[\left.\frac{x e^{-2\lambda x}}{2\lambda}\right]_0^\infty - \int_0^\infty \frac{e^{-2\lambda}}{-2\lambda}\right]$$

0

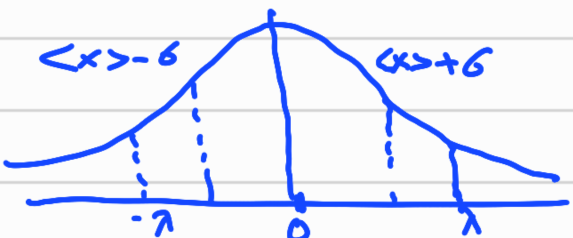
$$+\frac{\cancel{2}}{\cancel{2}\lambda}\int_0^\infty e^{-2\lambda x}\,dx = \frac{1}{\lambda}\left[\frac{e^{-2\lambda x}}{-2\lambda}\right]_0^\infty$$

$$\frac{1}{-2\lambda^2}\left[e^{-2\lambda x}\right]_0^\infty$$

$$\frac{1}{-2\lambda^2}\left[0 - e^0\right] = \frac{1}{-2\lambda^2}\left[0-1\right]$$

$$\langle x^2\rangle = +\frac{1}{2\lambda^2}$$

c) $\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{2\lambda^2} - 0} = \frac{1}{\sqrt{2}\,\lambda}$



Problem 1.7 Calculate $d\langle p\rangle/dt$. Answer:

momentum   potential energ

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle. \tag{1.38}$$

This is an instance of **Ehrenfest's theorem**, which asserts that *expectation values obey the classical laws.*[19]

$$\langle p\rangle = -i\hbar\int \psi^*\left(\frac{\partial\psi}{\partial x}\right)dx$$

$$\frac{d\langle p\rangle}{dt} = -i\hbar\int \frac{d}{dt}\left[\psi^*\left(\frac{\partial\psi}{\partial x}\right)\right]dx$$

$$= -i\hbar\int \frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x} + \psi^*\frac{\partial^2\psi}{\partial x^2}\quad dx$$

$$= -i\hbar\left(\int\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x} + \cancel{\psi^*\frac{\partial\psi}{\partial t}}\Big]_{-\infty}^{\infty} - \int\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial t}\,dx\right.$$

$u = \psi^*$ $dv = \frac{\partial^2\psi}{\partial x^2}$

$du = \frac{d\psi^*}{dx}dx$ $v = \frac{\partial\psi}{\partial t}$

$$= -i\hbar\int \frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial t}\frac{\partial\psi^*}{\partial x}\,dx$$

$$\frac{\partial\psi}{\partial t} = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right]$$

$$\frac{\partial\psi^*}{\partial t} = -\frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^*\right]$$

$$= -\cancel{i\hbar}\int \frac{-1}{\cancel{i\hbar}}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V\psi^*\right]\frac{\partial\psi}{\partial x} - \frac{1}{\cancel{i\hbar}}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right]\frac{\partial\psi^*}{\partial x}\,dx$$

$$\int \frac{-\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi^*}{\partial x} + V\psi^*\frac{\partial\psi}{\partial x} + V\psi\frac{\partial\psi^*}{\partial x}$$

$$\int \frac{-\hbar^2}{2m}\cancel{\left[\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial x^2}\frac{\partial\psi^*}{\partial x}\right]} + V\psi^*\frac{\partial\psi}{\partial t} + V\psi\frac{\partial\psi^*}{\partial x}$$

zero

$$\int V\psi^*\frac{\partial\psi}{\partial t} + V\psi\frac{\partial\psi^*}{\partial x} \, dx$$

$$u = V\psi^* \qquad dv = \frac{\partial\psi}{\partial t}$$

$$du = \frac{\partial V}{\partial x}\psi^* + V\frac{\partial\psi}{\partial x} \qquad v = \psi$$

$$\cancel{V\psi\psi^*} - \int \psi\left(\frac{\partial V}{\partial x}\psi^* + V\frac{\partial\psi}{\partial x}\right) + \int V\psi\frac{\partial\psi^*}{\partial x}$$

$$= \int -\psi\frac{\partial V}{\partial x}\psi^* - \cancel{\int V\psi\frac{\partial\psi}{\partial x}} + \cancel{\int V\psi\frac{\partial\psi}{\partial x}}$$

$$= \int -\psi\frac{\partial V}{\partial x}\psi^*$$

$$= \int \psi^*\frac{-\partial V}{\partial x}\psi \, dx = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

Problem 1.15 Show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \Psi_1^* \, \Psi_2 \, dx = 0$$

for any two (normalizable) solutions to the Schrödinger equation (with the same $V(x)$), Ψ_1 and Ψ_2.

$$\int_{-\infty}^{\infty} \frac{d}{dt} \psi_1^* \psi_2 \, dx = \int \frac{d\psi_1^*}{dt}\psi_2 + \frac{d\psi_2}{dt}\psi_1^* \, dx$$

$$\frac{d\psi}{dt} = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right] =$$

$$\frac{d\psi}{dt} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i}{\hbar}V\psi$$

$$\frac{d\psi^*}{dt} = \frac{-i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*$$

$$\int\left[\frac{-i\hbar}{2m}\frac{\partial^2\psi_1^*}{\partial x^2} + \cancel{\frac{i}{\hbar}V\psi_1^*}\right]\psi_2 + \left[\frac{i\hbar}{2m}\frac{\partial^2\psi_2}{\partial y^2} - \cancel{\frac{i}{\hbar}V\psi_2}\right]\psi_1^*$$

$$\int_{-\infty}^{\infty}\frac{i\hbar}{2m}\left[-\frac{\partial^2\psi_1^*}{\partial x}\psi_2 + \frac{\partial^2\psi_2}{\partial x}\psi_1^*\right] \rightarrow$$

نكمل بالاجزاء

و ستب : = Zero

$$= 0$$

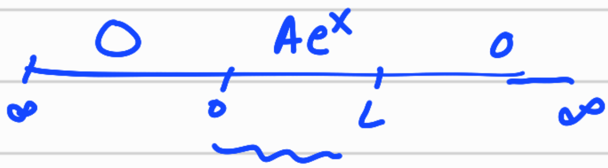
سؤال اضافي

suppose that it is determined that a particle has a wave function given by

$$\psi = \begin{cases} Ae^{x} & 0 \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

a) Normalize to evaluate A

$$\int_{-\infty}^{\infty} |\psi|^2 dx = 1$$



$$1 = \int_0^L A^2 e^{2x}\, dx = A^2 \int_0^L e^{2x}\, dx$$

$$1 = A^2 \frac{e^{2x}}{2}\Big]_0^L = \frac{A^2}{2} e^{2x}\Big]_0^L$$

$$1 = \frac{A^2}{2}\left(e^{2L} - 1\right) \qquad \frac{A^2 (e^{2L} - 1)}{2} = 1$$

$$A^2 = \frac{2}{e^{2L} - 1} \qquad A = \sqrt{\frac{2}{2^{L} - 1}}$$

سؤال اضافي

(2)

a) Normalize the function $e^{i(kx-\omega t)}$ in the region for 0 to $3a$

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = \int_0^{3a} \psi \psi^* dx$$

$$1 = \int A e^{i(kx-\omega t)} A e^{-i(kx-\omega t)} dx$$

$$1 = A^2 \int_0^{3a} 1 \, dx$$

$$1 = A^2 \left[x \right]_0^{3a} \qquad 1 = A^2 (3a)$$

$$A^2 = \frac{1}{3a} \qquad A = \sqrt{\frac{1}{3a}}$$

b) find $\langle x \rangle$ $\langle x^2 \rangle$ σ

$$\langle x \rangle = \int x |\psi|^2 dx$$

$$\langle x \rangle = \int x \frac{1}{3a} e^{-i(\cancel{kx-\omega t})} e^{i(\cancel{kx-\omega t})} dx$$

$$\langle x \rangle = \frac{1}{3a} \int_0^{3a} x = \frac{1}{3a} \left. \frac{x^2}{2} \right]_0^{3a}$$

$$\langle x \rangle = \frac{1}{3a}\left[\frac{9a^2 - 0}{2}\right] = \frac{3a}{2}$$

$$\langle x^2 \rangle = \frac{1}{3a}\int_0^{3a} x^2 = \frac{1}{3a}\,\frac{x^3}{3}\Big]_0^{3a}$$

$$\langle x^2 \rangle = \frac{1}{3a}\left[\frac{(3a)^3}{3} - 0\right]$$

$$\langle x^2 \rangle = 3a^2$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma = \sqrt{3a^2 - \frac{9}{4}a^2}$$

$$\sigma = \sqrt{\frac{12}{4}a^2 - \frac{9}{4}a^2} = \sqrt{\frac{3a^2}{4}}$$

Problem 1.9 A particle of mass m has the wave function

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]},$$

where A and a are positive real constants.

(a) Find A.

(b) For what potential energy function, $V(x)$, is this a solution to the Schrödinger equation?

(c) Calculate the expectation values of x, x^2, p, and p^2.

(d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

$$|\psi|^2 = \psi\psi^* = Ae^{-a[mx^2/\hbar + \cancel{it}]} \cdot Ae^{-a[mx^2/\hbar - \cancel{it}]}$$

$$|\psi|^2 = A^2 e^{-2amx^2/\hbar}$$

$$\int_0^\infty e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

1) find A

$$1 = \int_{-\infty}^{\infty} |\psi|^2 = A^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar}\, dx$$

$$= 2A^2 \int_0^\infty e^{-x^2\left(\frac{2am}{\hbar}\right)} = \cancel{2}A^2 \frac{1}{\cancel{2}}\sqrt{\frac{\pi\hbar}{2am}}$$

$$1 = A^2\sqrt{\frac{\pi\hbar}{2am}}$$

$$A^2 = \sqrt{\frac{2am}{\pi\hbar}} \qquad A = \sqrt[4]{\frac{2am}{\pi\hbar}}$$

$$\psi = \sqrt[4]{\frac{2am}{\pi\hbar}}\, e^{-a[mx^2/\hbar + it]}$$

✓

b) تطبیق معادله Sh و تعیین v

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V \psi$$

$$V = \frac{\hbar}{i\psi}\left[\frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{\partial \psi}{\partial t}\right]$$

$$V = \frac{\hbar\, e^{a[mx^2/\hbar + it]}}{i\sqrt[4]{\frac{2am}{\pi\hbar}}}\left[\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\left[\sqrt[4]{\frac{2am}{\hbar\pi}}\, e^{-a\left[\frac{mx^2}{\hbar}+it\right]}\right] - \frac{\partial}{\partial t}\sqrt[4]{\frac{2am}{\hbar\pi}}\, e^{-a\left[\frac{mx^2}{\hbar}+it\right]}\right]$$

$$V = \frac{\hbar}{i}\sqrt[4]{\frac{\pi\hbar}{2am}}\, e^{a\left[\frac{mx^2}{\hbar}+it\right]}\left[\frac{i\hbar}{2m}\sqrt[4]{\frac{2am}{\hbar\pi}}\frac{\partial}{\partial x}\, e^{-a\left[\frac{mx^2}{\hbar}+it\right]}\left[\frac{-a2mx}{\hbar}\right] - \sqrt[4]{\frac{2am}{\hbar\pi}}\, e^{-a\left[\frac{mx^2}{\hbar}+it\right]}(-ai)\right.$$

$$V = \frac{\hbar}{i} e^{a\left[\frac{mx^2}{\hbar}+i\hbar\right]}\left[\frac{i\hbar}{2m}\left[\frac{-2^2m^2a^2x^2}{\hbar^2}\right] e^{-a\left[\frac{mx^2}{\hbar}-it\right]} + \left[\frac{-a2m}{\hbar}\right] e^{-a\left[\frac{mx^2}{\hbar}-i\hbar\right]}\right.$$

$$+\, ai\, e^{-a\left[\frac{mx^2}{\hbar}+it\right]}$$

$$V = \hbar\, \cancel{e^{a\left[\frac{mx^2}{\hbar}+i\hbar\right]}}\left[\frac{\cancel{\hbar}}{2\cancel{m}}\frac{2\cancel{m}a}{\hbar}\left[-1 + \frac{2ma}{\hbar}x^2\right] \cancel{e^{-a\left[\frac{mx^2}{\hbar}-i\hbar\right]}}\right]$$

$$+\, a\, \cancel{e^{-a\left[\frac{mx^2}{\hbar}+it\right]}}$$

$$V = \hbar\left(\left[a\left(-1 + \frac{2max^2}{\hbar}\right)\right] + a\right)$$

$$V = \hbar\left(-\cancel{a} + \frac{2ma^2x^2}{\hbar} + \cancel{a}\right) = 2ma^2x^2$$

c) $\langle x \rangle = \int x|\psi|^2$

$$\langle x \rangle = \int \cancel{x}\ A^2 e^{-2amx^2/\hbar}\ dx$$

$$\langle x \rangle = \sqrt{\frac{2am}{\pi\hbar}} \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar}\ dx \quad \checkmark$$

$$\langle x \rangle = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi|^2 dy$$

$$\int_0^{\infty} x^2 e^{-ax^2} = \frac{1}{4a}\left(\frac{\pi}{a}\right)^{1/2}.$$

$$\langle x^2 \rangle = \sqrt{\frac{2am}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-2amx^2/\hbar}\ dx$$

$$\langle x^2 \rangle = 2\sqrt{\frac{2am}{\pi\hbar}} \int_0^{\infty} x^2 e^{-x^2\left(\frac{2am}{\hbar}\right)}$$

$$\langle x^2 \rangle = \cancel{2}\sqrt{\cancel{\frac{2am}{\pi\hbar}}}\left[\frac{1}{4\left(\frac{\cancel{2}am}{\hbar}\right)} \frac{\sqrt{\cancel{\pi}}}{\cancel{\sqrt{\frac{2am}{\hbar}}}}\right] = \frac{\hbar}{4ma}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*\left(i\hbar\frac{\partial}{\partial x}\right)\psi$$

$$\langle p \rangle = i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x} \, dx$$

$$\langle p \rangle = i\hbar \int_{-\infty}^{\infty} \sqrt[4]{\frac{2am}{\hbar}} e^{-a\left[\frac{mx^2}{\hbar} - it\right]} \frac{\partial}{\partial x} \sqrt[4]{\frac{2am}{\hbar}} e^{-a\left[\frac{mx^2}{\hbar} + it\right]} dx$$

$$\langle p \rangle = i\hbar \sqrt{\frac{2am}{\hbar}} \int e^{-a\left[\frac{mx^2}{\hbar} - it\right]} \cdot e^{-a\left[\frac{mx^2}{\hbar} + it\right]} \left(\frac{-a2mx}{\hbar}\right)$$

$$\langle p \rangle = i\hbar \sqrt{\frac{2am}{\hbar}} \left(-\frac{2am}{\hbar}\right) \underbrace{\int_{-\infty}^{\infty} x e^{-2a\frac{mx^2}{\hbar}}}_{0}$$

$$\langle p \rangle = 0$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \psi \, dx$$

$$= \hbar a m$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{4ma} - 0} = \sqrt{\frac{\hbar}{4ma}}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar a m - 0} = \sqrt{\hbar a m}$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

$$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{4\cancel{ma}}} \sqrt{\hbar \cancel{am}} = \frac{\sqrt{\hbar}\sqrt{\hbar}}{\sqrt{4}} = \frac{\hbar}{2}$$

Problem 1.16 A particle is represented (at time $t = 0$) by the wave function

$$\Psi(x,0) = \begin{cases} A\left(a^2 - x^2\right), & -a \le x \le +a, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the normalization constant A.
(b) What is the expectation value of x?
(c) What is the expectation value of p? (Note that you *cannot* get it from $\langle p \rangle = md\langle x \rangle/dt$. Why not?)

a) $1 = \int_{-\infty}^{\infty} |\psi^2| \, dx = \int_{-a}^{a} A^2 (a^2 - x^2)^2 dx$

$$1 = 2A^2 \int_0^a a^4 - 2a^2x^2 + x^4 \quad dx$$

$$1 = 2A^2 \left[a^4 x - \frac{2}{3} a^2 x^3 + \frac{x^5}{5} \right]_0^a = 2A^2 \left[a^5 - \frac{2}{3} a^5 + \frac{a^5}{5} \right]$$

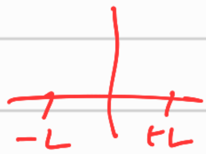
$$1 = 2A^2 \left[\frac{15a^5}{15} - \frac{10a^5}{15} + \frac{3a^5}{15} \right] \Rightarrow 1 = 2A^2 \, \frac{8a^5}{15}$$

$$1 = \frac{16}{15} A^2 a^5$$

$$A^2 = \frac{15}{16a^5} \qquad A = \sqrt{\frac{15}{16a^5}}$$

b) $\langle x \rangle = \int x |\psi^2| dx = A^2 \int_{-a}^{a} x \, (a^2 - x^2)^2 \, dx$

$$= \frac{15}{16} a^5 \int_{-a}^{a} (x)(a^4 + 2a^2x^2 + x^4) \, dx$$



odd

$= 0$

c) $\langle p \rangle = \int_{}^{\infty} \psi^{*} \left(-i\hbar \frac{\partial}{\partial x}\right) \psi$

$$\langle p \rangle = -i\hbar A^2 \int_{-a}^{a} (a^2 - x^2) \frac{\partial}{\partial x}(a^2 - x^2) \, dx$$

$$\langle p \rangle = -i\hbar A^2 \int (a^2 - x^2)(-2x) \, d$$

$$\langle p \rangle = +2\hbar A^2 \int_{-a}^{a} x(a^2 - x^2)$$

even

$= \text{zero}$

(d) Find the expectation value of x^2.
(e) Find the expectation value of p^2.
(f) Find the uncertainty in x (σ_x). $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$
(g) Find the uncertainty in p (σ_p). $\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$
(h) Check that your results are consistent with the uncertainty principle.

c) $\langle x^2 \rangle = \int x^2 |\psi^2| \, dx$

$$\frac{15}{16a^5} \int_{-a}^{a} x^2 (a^2 - x^2)^2 \, dx$$

$$\frac{15}{16a^5} \int_{-a}^{a} (x^2 a^4 - 2a^2 x^4 + x^6) \, dx$$

$$2 \frac{15}{16a^5} \int_{0}^{a} x^2 a^4 - 2a^2 x^4 + x^6$$

$$\frac{15}{8a^5}\left[\frac{x^3}{3}a^4 - 2a^2\frac{x^5}{5} + \frac{x^7}{7}\right]_0^a$$

$$\frac{15}{8a^5}\left[\frac{a^7}{3} - \frac{2a^7}{5} + \frac{a^7}{7}\right]$$

$$\frac{15}{8}a^7\left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right]$$

$$\frac{a^7}{8}\left[5 - 6 + \frac{15}{7}\right]$$

$$\frac{a^7}{8}\left[\frac{\cancel{-7}}{7} + \frac{15}{7}\right] = \frac{a^7\cancel{8}}{\cancel{8}\times 7} = \frac{a^7}{7}$$

e) $\langle p^2 \rangle = \int_{-a}^{a} \psi(-i\hbar)^2 \frac{\partial^2}{\partial x^2}\psi$ $\qquad (i)^2 = -1$

$$\langle p^2 \rangle = \hbar^2 \int_{-a}^{a} A(a^2-x^2)\frac{\partial^2}{\partial x^2}A\underbrace{(a^2-x^2)}_{0-2x}$$

$$\langle p^2 \rangle = \hbar^2 A^2 \int_{-a}^{a}(a^2-x^2)(-2)$$

$$\langle p^2 \rangle = (-2)\,2\hbar^2\frac{15}{16a^5}\int_0^a (a^2-x^2)\,dx$$

$$= -\hbar^2\frac{15}{4a^5}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= -\frac{15\hbar^2}{4a^5}\left[a^3 - \frac{a^3}{3}\right]$$

$$\langle p^2 \rangle = -\frac{15\hbar^2}{4a^5}\,\frac{2a^3}{3} = -\frac{5\hbar^2}{2a^2}$$

f) $$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\sigma_x = \sqrt{\frac{a^2}{7}}$$

g) $$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \sqrt{\frac{5\hbar^2}{2a^2}}$$

h) $$\sigma_x \sigma_x \geq \frac{\hbar}{2}$$

$$\sqrt{\frac{a^2}{7}}\sqrt{\frac{5\hbar^2}{2a^2}} \overset{?}{\geq} \frac{\hbar}{2} \rightarrow 0.5\hbar$$

$$\sqrt{\frac{\cancel{a^2}\,5\,\hbar^2}{14\,\cancel{a^2}}} = \sqrt{\frac{5}{14}}\,\hbar = 0.59\hbar$$

ملخص قوانين شابتر ١

$$\Rightarrow \int \psi^* \psi \, dx = \int |\psi|^2 dx = 1$$

$$\Rightarrow \langle x \rangle = \int x \, |\psi^2| \, dx$$

$$\Rightarrow \langle x^2 \rangle = \int x^2 \, |\psi^2| \, dx$$

$$\Rightarrow \langle p \rangle = \int \psi^* \, i\hbar \frac{\partial}{\partial x} \psi \, dx$$

$$\Rightarrow \langle p^2 \rangle = \int \psi \left(i\hbar \frac{\partial}{\partial x}\right)^2 \psi \, dx$$

$$\Rightarrow \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

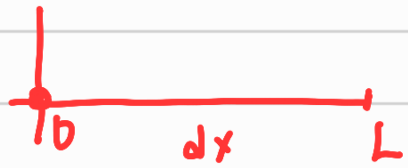
$$\Rightarrow \sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$\Rightarrow \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

اسئلة الواجب من خارج الكتاب

- What are the dimensions of
    1. The wave function in one dimension $\psi(\boldsymbol{x})$
    2. The wave function in three dimensions $\psi(\boldsymbol{r})$
    3. The wavefunction in the reciprocal space $\psi(\boldsymbol{k})$

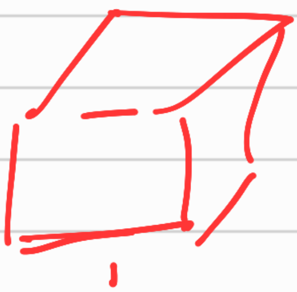
1)



$$|\psi|^2 = \frac{1}{L}$$

$$\psi = \frac{1}{\sqrt{L}} = L^{-1/2}$$

2)



$$|\psi|^2 = \frac{1}{L^3}$$

$$\psi = \sqrt{\frac{1}{L^3}} = \frac{1}{L^{3/2}} = L^{-3/2}$$

3) reciprocal $\psi(k)$

$\psi = L^{-3/2}$ 3d space

$\psi = L^{-1/2}$ 1d space

The wavefunction for a particle in one dimension is given by,

$$|\psi_1(x)| = A_1 e^{-y^2/4}$$

Another state that the particle might be in is,

$$|\psi_2(x)| = A_2 e^{-y^2/8}$$

A third state the particle may be in,

$$|\psi_3(x)| = A_3(|\psi_1(x)| + |\psi_2(x)|)$$

1. Normalize all three states in the interval $-\infty < y < \infty$
2. Is the probability of finding the particle in the interval $0 < y < 1$ the same as the sum of the separate probabilities for the states $\psi_1(x)$ and $\psi_2(x)$
3. Repeat part (2) for the interval $-1 < y < 1$

$$\int_0^\infty e^{-ax^2} dx$$

ψ_1

(1) $$1 = \int_{-\infty}^{\infty} |\psi_1|^2 dy = A_1^2 \int_{-\infty}^{\infty} \left(e^{-\frac{y^2}{4}}\right)^2 dy = A_1^2 \int_{-\infty}^{\infty} e^{-\frac{2y^2}{4}}$$

$$1 = A_1^2 \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} = A_1^2 \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$

$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$a = -\frac{1}{2}$

$$1 = 2A_1^2 \int_0^\infty e^{-y^2/2} = \cancel{2} A_1^2 \left(\frac{1}{\cancel{2}} \sqrt{\frac{\pi}{1/2}}\right)$$

$$1 = A_1^2 \sqrt{2\pi}$$

$$A_1^2 = \frac{1}{\sqrt{2\pi}}$$

$$A_1 = \frac{1}{\sqrt[4]{2\pi}}$$

ψ_2

$$1 = \int_{-\infty}^{\infty} A_2^2 \left(e^{y^2/8}\right)^2 = A_2^2 \int_{-\infty}^{\infty} e^{\frac{y^2}{4}} dy$$

$$1 = 2A^2 \int_0^\infty e^{y^2/4} dy = \cancel{2} A_2^2 \left(\frac{1}{\cancel{2}} \sqrt{\frac{\pi}{1/4}}\right)$$

$$1 = A_2^2 \sqrt{4\pi} \qquad A_2^2 = \frac{1}{\sqrt{4\pi}}$$

$$A_2 = \frac{1}{\sqrt[4]{4\pi}}$$

$$\Psi_3 \Rightarrow A_3\left(A_1 e^{-y^2/4} + A_2 e^{-y^2/8}\right)$$

$$1 = \int |\Psi_3|^2 = A_3^2 \int \left(A_1 e^{-y^2/4} + A_2 e^{-y^2/8}\right)^2$$

$$1 = A_3^2 \int A_1^2 e^{-y^2/2} + 2A_1A_2 e^{-3\frac{y^2}{8}} + A_2^2 e^{-y^2/4}$$

(1) (2) (3)

(1) $\int_{-\infty}^{\infty} A_1^2 e^{-y^2/2} = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-y^2/2} = \frac{2}{\sqrt{2\pi}} \frac{1}{2}\sqrt{2\pi} = 1$

(3) $\int_{-\infty}^{\infty} A_2^2 e^{-\frac{y^2}{4}} = \frac{2}{\sqrt{4\pi}} \int_0^{\infty} e^{-\frac{y^2}{4}} = \frac{\cancel{2}}{\cancel{\sqrt{4\pi}}} \frac{1}{2} \cancel{\sqrt{4\pi}} = 1$

(2) $2A_1A_2 \int_{-\infty}^{\infty} e^{-3\frac{y^2}{8}} = 2 \frac{1}{\sqrt[4]{2\pi}} \cdot \frac{2}{\sqrt[4]{4\pi}} \int_0^{\infty} e^{-\frac{3y^2}{8}}$

$$= \frac{4}{\sqrt[4]{8\pi^2}} \frac{1}{2} \sqrt{\frac{\pi 8}{3}} = \frac{2}{8^{1/4} \cancel{\pi^{1/2}}} \cancel{\pi^{1/2}} \frac{8^{1/2}}{3^{1/2}}$$

$\sqrt{\pi} = \pi^{1/2}$

$\sqrt[4]{\pi} = \pi^{1/4}$

$\sqrt[4]{\pi^2} = \pi^{2/4}$

$$\frac{2\,8^{1/2}}{3^{1/2}} = 1.94$$

$$1 = A_3^2\,[1 + 1.94 + 1]$$

$$1 = A_3^2\,[3.94]$$

$$A_3^2 = \frac{1}{3.94} \qquad A_3 = \frac{1}{\sqrt{3.94}} = 0.503$$

b) $$\int_0^1 \psi_1^2 = \int_0^1 \left(A_1 e^{-y^2/4}\right)^2 = A_1^2 \int_0^1 e^{-y^2/2}\,dy$$

$P_{\psi_1}\ 0>x>1$

0.532

$$\int_0^1 \psi_2^2 = \int_0^1 \left(A_2 e^{-y^2/8}\right)^2 = A_2^2 \int_0^1 e^{-y^2/4}\,dy$$

$P_{\psi_2}\ 0>x>1$

0.548

$$A_3^2\int_0^1 (\psi_1 + \psi_2)^2 = A_3^2\left[\int_0^1 \psi_1^2 + \int_0^1 \psi_2^2 + 2\int_0^1 \psi_1\psi_2\right]$$

0.27

$$P_{\psi_3}\ 0>x>1 = A_3^2\left[A_1^2\int_0^1 e^{-y^2/2} + A_2^2\int_0^1 e^{-y^2/4} + 2A_1A_2\int e^{-\frac{3y^2}{8}}\right]$$

$$\underline{P_{\psi_3}\ 0>x>1} = A_3\left[\underline{P_{\psi_1} + P_{\psi_2}} + 2A_1A_2\int_0^1 e^{-3y^2/8}\right]$$

$$P\psi_1 + P\psi_2 = P\psi_3$$

X

The sum of Probality for ψ_1 and ψ_2

is not equal to the probility of ψ_3

c) $$P_{\psi_1} = \int_{-1}^{1} A_1^2 e^{-y^2/2}$$

$$P_{\psi_2} = \int_{-1}^{1} A_2^2 \, e^{-y^2/4}$$

$$P_{\psi_3} = A_3^2 \int_{-1}^{1} (\psi_1 + \psi_2)^2$$

$$P_{\psi_3} = A_3^3 \int_{-1}^{1} \psi_1^2 + \psi_2^2 + 2\psi_1\psi_2$$

$$= A_3^3 \Big[\underbrace{\int A_1^2 e^{-y^2/2}}_{P_{\psi_1}} + \underbrace{\int A_2^2 e^{-y^2/4}}_{P_{\psi_2}} + A_1 A_2 \int e^{-3y^2/8}$$

$$P_{\psi_3} \neq P_{\psi_1} + P_{\psi_2}$$