Chapter 3
KINEMATICS OF FLUID FLOW

1. Types of Fluid Flow.
- 1.1 Real -or -Ideal fluid.
- 1.2 Uniform -or -Non-uniform Flows.
- 1.3 One, Two -or -Three Dimensional Flows.
- 1.4 Rotational -or -Irrotational Flows.

2. Circulation -or -Vorticity.

3. Stream Lines, Flow Field and Stream Tube.
4. Velocity and Acceleration in Flow Field.
5. Continuity Equation for One Dimensional Steady Flow.
6. Stream Function & Velocity Potential
7. Reynold’s Number

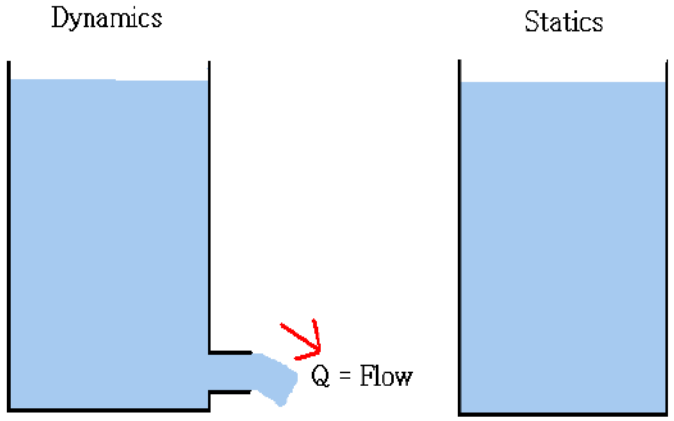
Fluid mechanics

Is a study of the behavior of fluids, either at rest (fluid statics) or in motion (fluid dynamics).

- We now turn our attention to fluids in motion.
- Instead of trying to study the motion of each particle of the fluid as a function of time.
- We describe the properties of a moving fluid at each point as a function of time.

Fluid Dynamics

The laws of Statics that we have learned cannot solve Dynamic Problems There is no way to solve for the flow rate, or Q. Therefore, we need a new dynamic approach to Fluid Mechanics.



- **Fluid Kinematics:** deals with the motion of fluids without considering the forces and moments which create the motion.

We define field variables which are functions of space and time

Pressure field, $\vec{P} = \vec{P}(x, y, z, t)$

Velocity field $\vec{V} = \vec{V}(x, y, z, t)$

$$\vec{V} = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$

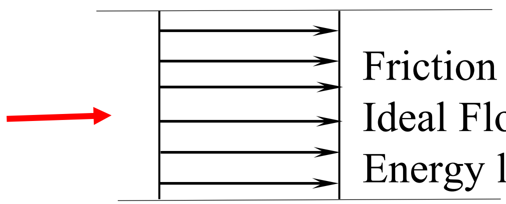
Acceleration field $\vec{a} = \vec{a}(x, y, z, t)$

$$\vec{a} = a_x(x, y, z, t)\vec{i} + a_y(x, y, z, t)\vec{j} + a_z(x, y, z, t)\vec{k}$$

Types of fluid Flow

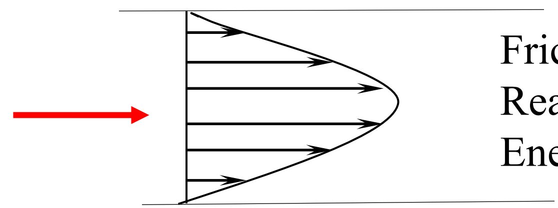
1. Real and Ideal Flow:

If the fluid is considered frictionless with zero viscosity it is called ideal.
In real fluids the viscosity is considered and shear stresses occur causing conversion of mechanical energy into thermal energy



Friction = 0
Ideal Flow ($\mu = 0$)
Energy loss =0

Ideal



Friction $\neq$ o
Real Flow ($\mu \neq 0$)
Energy loss $\neq 0$

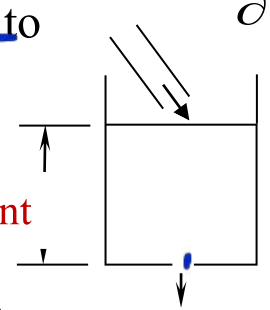
Real

2. Steady and Unsteady Flow

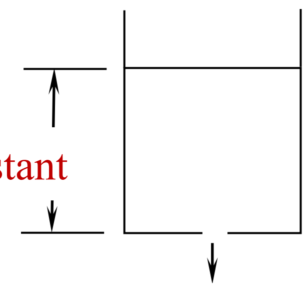
$$\frac{\partial(\ )}{\partial(t)} = 0 \text{ steady}$$

$$\frac{\partial(\ )}{\partial(t)} \neq 0 \text{ unsteady}$$

Steady flow occurs when conditions of a point in a flow field don't change with respect to time (v, p, H…..changes w.r.t. time)



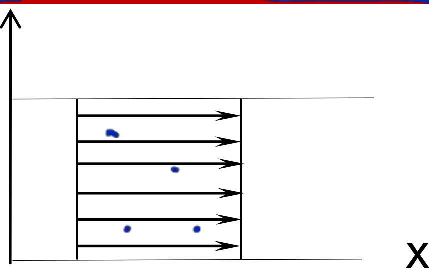
Steady Flow with respect to time
•Velocity is constant at certain position w.r.t. time

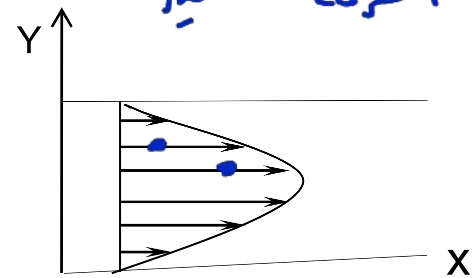


Unsteady Flow with respect to time. Velocity changes at certain position w.r.t. time

uniform ⇐ السرعة ثابتة في جميع الاماكن وجميع الاوقات
nonunifor ← السرعة متغيرة

3. Uniform and Non uniform Flow





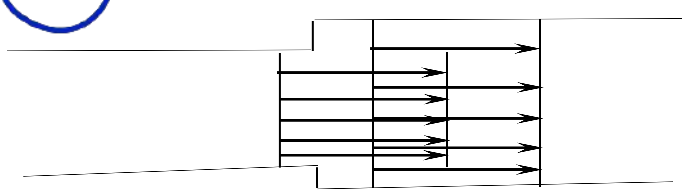
Uniform Flow means that the velocity is constant at certain time in different positions (doesn't depend on any dimension x or y or z)

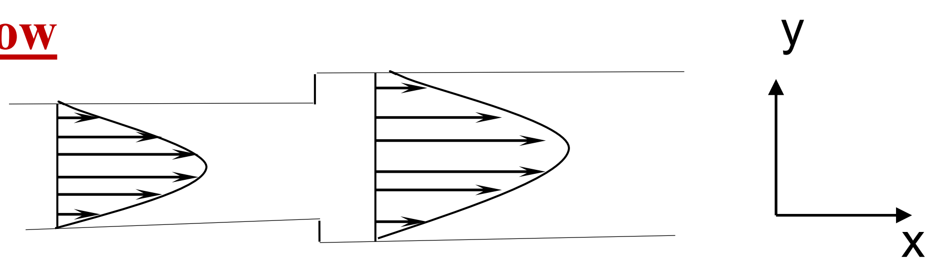
$$\frac{\partial(\ )}{\partial(x)} = 0 \quad \text{uniform}$$

$$\frac{\partial(\ )}{\partial(x)} \neq 0 \quad \text{Non-uniform}$$

Non- uniform Flow means velocity changes at certain time in different positions (depends on dimension x or y or z)

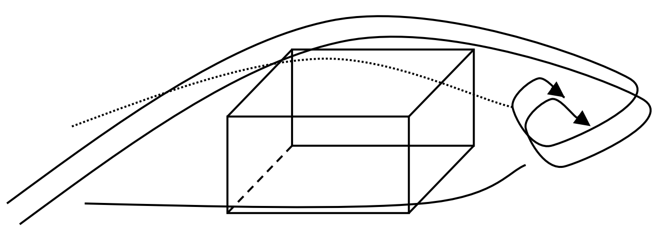
One,Two and three Dimensional Flow





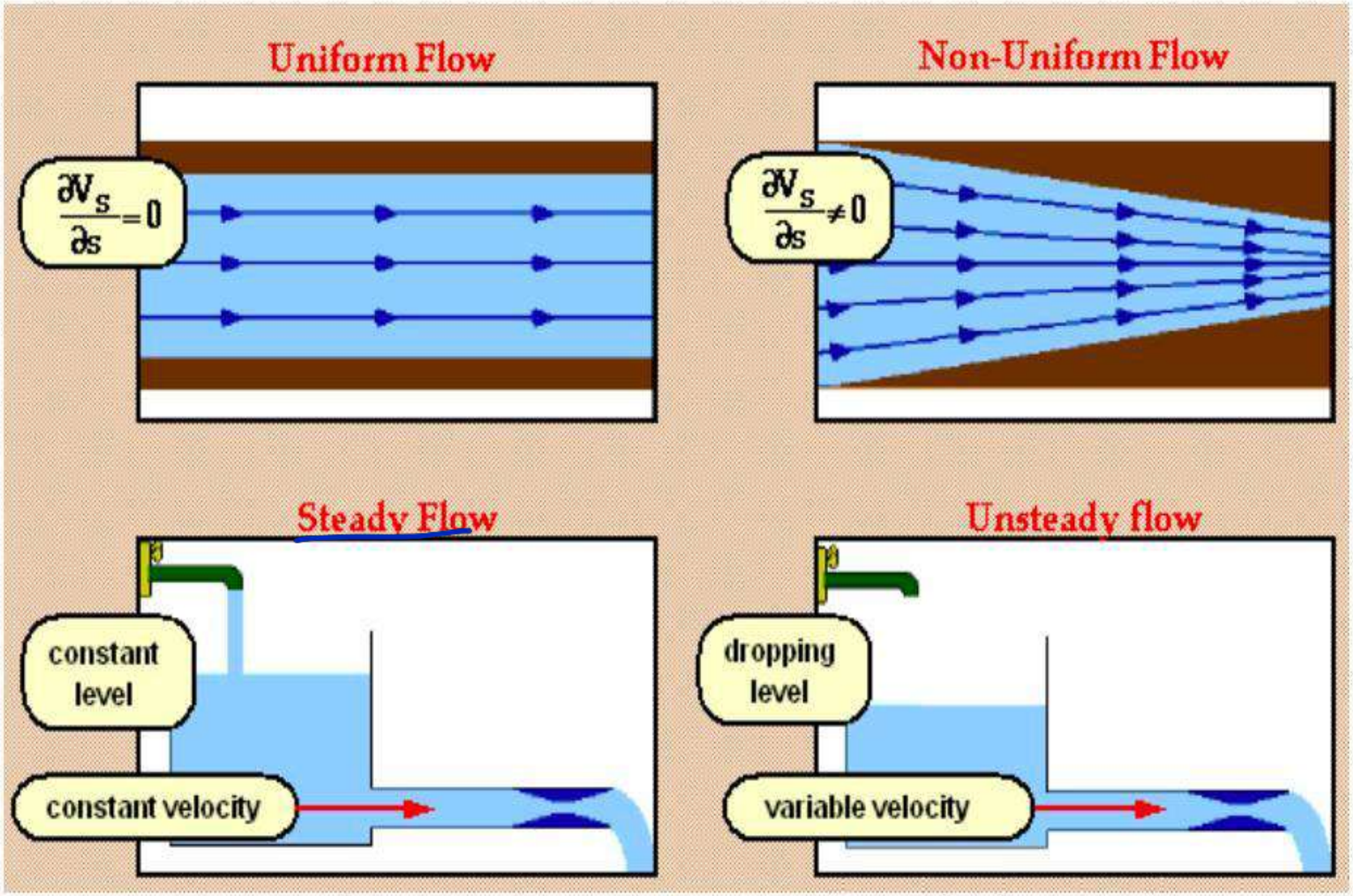
One dimensional flow means that the flow velocity is function of one coordinate
$V = f(X \text{ or } Y \text{ or } Z)$

Two dimensional flow means that the flow velocity is function of two coordinates
$V = f(X,Y \text{ or } X,Z \text{ or } Y,Z)$



Three dimensional flow means that the flow velocity is function of there coordinates $V = f(X,Y,Z)$

Summary



$$\frac{\partial(\,)}{\partial t} = 0 \qquad \frac{\partial(\,)}{\partial t} \neq 0$$

Uniform Flow, Steady Flow

- **Uniform flow:** flow velocity is the same magnitude and direction at every point in the fluid.
- **Non-uniform:** If at a given instant, the velocity is not the same at every point the flow. (In practice, by this definition, every fluid that flows near a solid boundary will be non-uniform - as the fluid at the boundary must take the speed of the boundary, usually zero. However if the size and shape of the cross-section of the stream of fluid is constant the flow is considered uniform.)
- **Steady:** A steady flow is one in which the conditions (velocity, pressure and cross-section) may differ from point to point but DO NOT change with time.
- **Unsteady:** If at any point in the fluid, the conditions change with time, the flow is described as unsteady. (In practice there is always slight variations in velocity and pressure, but if the average values are constant, the flow is considered steady.)

➢ **الجريان المنتظم (Uniform flow):**

هو الجريان الذي تكون فيه سرعة السريان (المقدار والاتجاه) متماثلة في كل نقطة داخل السائل.

➢ **الجريان غير المنتظم (Non-uniform):**

هو الجريان الذي لا تكون فيه السرعة متماثلة في جميع النقاط عند لحظة معينة. (عمليًا، ووفقًا لهذا التعريف، فإن كل سائل ينساب بالقرب من سطح صلب سيكون غير منتظم – لأن السائل عند السطح يجب أن يأخذ سرعة السطح، والتي تكون عادةً صفرًا. ولكن إذا كان حجم وشكل المقطع العرضي للتيار السائل ثابتًا، فيُعتبر الجريان منتظمًا.)

➢ **الجريان المستقر (Steady):**

هو الجريان الذي قد تختلف فيه الظروف (مثل السرعة والضغط والمقطع العرضي) من نقطة إلى أخرى، لكنها **لا تتغير مع الزمن**.

➢ **الجريان غير المستقر (Unsteady):**

هو الجريان الذي تتغير فيه الظروف (كالسرعة والضغط) مع الزمن عند أي نقطة في السائل. (عمليًا، هناك دائمًا بعض التغيرات الطفيفة في السرعة والضغط، ولكن إذا كانت القيم المتوسطة ثابتة، فإن الجريان يُعتبر مستقرًا.)

النوع	الاسم بالإنجليزية	التعريف بالعربية
منتظم	Uniform flow	هو الجريان الذي تكون فيه سرعة السريان (المقدار والاتجاه) متماثلة في جميع النقاط داخل السائل.
غير منتظم	Non-uniform flow	هو الجريان الذي تختلف فيه السرعة من نقطة لأخرى عند لحظة معينة. يحدث عادةً قرب الأسطح الصلبة، لكن إذا كان المقطع العرضي ثابتًا يُعتبر الجريان منتظمًا.
مستقر	Steady flow	هو الجريان الذي قد تختلف فيه الظروف من نقطة لأخرى ولكنها لا تتغير مع مرور الزمن.
غير مستقر	Unsteady flow	هو الجريان الذي تتغير فيه الظروف (مثل السرعة أو الضغط) مع الزمن في نقطة معينة.

الجريان المستقر المنتظم (Steady uniform flow):

- الشروط: لا تتغير لا مع الموقع في التيار ولا مع الزمن.
- مثال: جريان الماء في أنبوب ذو قطر ثابت وسرعة ثابتة.

الجريان المستقر غير المنتظم (Steady non-uniform flow):

- الشروط: تتغير من نقطة لأخرى في التيار، لكنها لا تتغير مع الزمن.
- مثال: الجريان في أنبوب متناقص القطر مع سرعة ثابتة عند المدخل – حيث تتغير السرعة كلما تحركت على طول الأنبوب نحو المخرج.

الجريان غير المستقر المنتظم (Unsteady uniform flow):

- عند لحظة زمنية معينة، تكون الشروط متماثلة في كل نقطة، لكنها تتغير مع الزمن.
- مثال: أنبوب ذو قطر ثابت متصل بمضخة تضخ بمعدل ثابت، ثم يتم إيقاف تشغيل المضخة.

الجريان غير المستقر غير المنتظم (Unsteady non-uniform flow):

- جميع شروط الجريان قد تتغير من نقطة لأخرى ومع الزمن في كل نقطة.
- مثال: الأمواج في قناة.

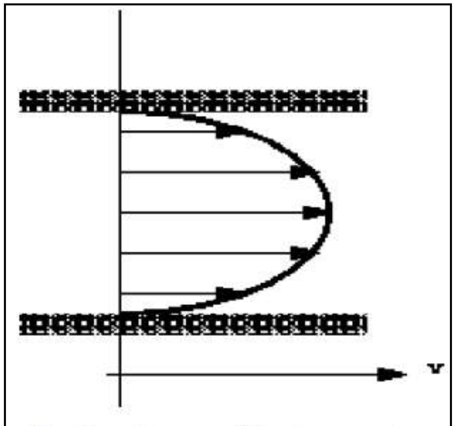
Uniform Flow, Steady Flow

- **Steady uniform flow:**
  - Conditions: do not change with position in the stream or with time.
  - Example: the flow of water in a pipe of constant diameter at constant velocity.
- **Steady non-uniform flow:**
  - Conditions: change from point to point in the stream but does not change with time.
  - Example: flow in a tapering pipe with constant velocity at the inlet- velocity will change as you move along the length of the pipe toward the exit.
- **Unsteady uniform flow:**
  - At a given instant in time the conditions at every point are the same, but will change with time.
  - Example: a pipe of constant diameter connected to a pump pumping at a constant rate which is then switched off.
- **Unsteady non-uniform flow:**
  - Every condition of the flow may change from point to point and with time at every point.
  - Example: waves in a channel.

Velocity Profile

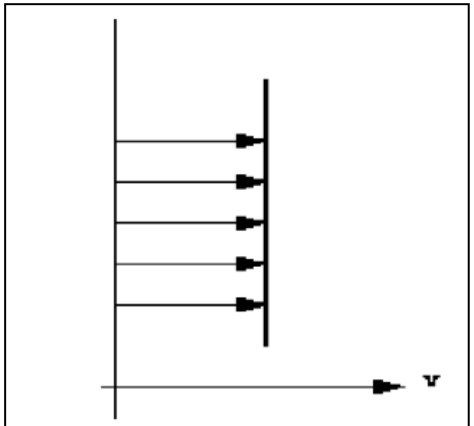
Flow in a pipe

Consider the flow in a pipe in which water is flowing. At the pipe wall the velocity of the water will be zero. The velocity will increase as we move toward the center of the pipe. This change in velocity across the direction of flow is known as ***velocity profile.***



Velocity profile in a pipe.

Because particles of fluids next to each other are moving with different velocities there are shear forces in the moving fluid i.e shear forces are **normally** present in a moving fluid. On the other hand, if a fluid is a long way from the boundary and all the particles are travelling with the same velocity, the velocity profile would look something like this:

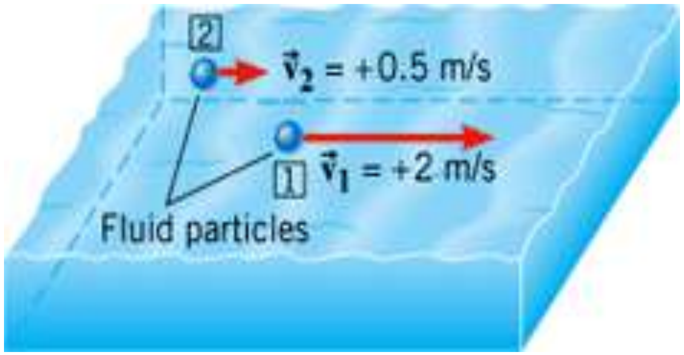


Velocity profile in uniform f

- And there will be no shear forces present as all particles have zero relative velocity. In practice we are concerned with flow past solid boundaries; airplanes, cars, pipe walls, river channels etc. and **shear forces will be present.**

➢ **Fluid flow can be steady or unsteady.**

Two fluid particles in a stream. At different locations in the stream the particle velocities may be different, as indicated by V_1 and V_2.



- In ***steady flow*** the **velocity** of the fluid particles at any point is **constant** as time passes.
- In ***Unsteady flow*** It exists whenever the **velocity** at a point in the fluid **changes** as time passes.

➢ Flow Characteristics

When fluid is in motion, its flow can be characterized as being one of two main types:

1. **Steady, or laminar**, if each particle: of the fluid follows a smooth path, such that the paths of different particles never cross each other, as shown in figure 3.

Figure3. Laminar flow around an automobile in a test wind tunnel

- In steady flow, the velocity of the fluid at any point remains constant in time.

2. Above a certain critical speed, fluid flow becomes turbulent. **Turbulent flow** is irregular flow characterized by small wind pool-like regions, (figure 4).

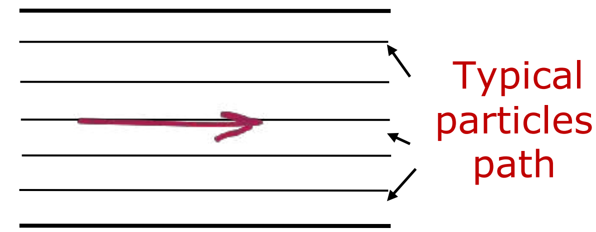
The smoke first moves in laminar flow at the bottom and then in turbulent flow above

Figure 4. Hot gases from a cigarette made visible by smoke particles.

Laminar and Turbulent Flow

Laminar flow:

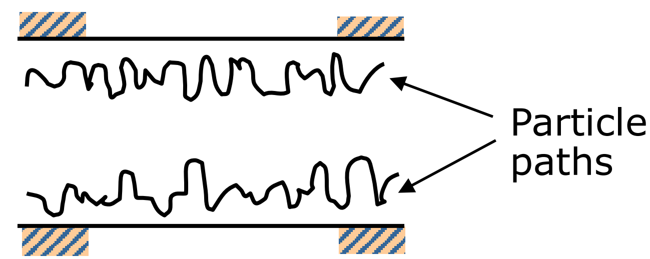
- all the particles proceed along smooth parallel paths and all particles on any path will follow it without deviation.
- Hence all particles have a velocity only in the direction of flow.



Laminar flow

Turbulent Flow:

- the particles move in an irregular manner through the flow field.
- Each particle has superimposed on its mean velocity fluctuating velocity components both transverse to and in the direction of the net flow.



Turbulent flow

Transition Flow:

- exists between laminar and turbulent flow.
- In this region, the flow is very unpredictable and often changeable back and forth between laminar and turbulent states.

(a) Laminar flow of the Firehole River at Yellowstone;
(b) transition from laminar to turbulent flow in rising smoke;
(c) turbulent flow at the Upper Falls on the Yellowstone River.

a) b) c)

- *Turbulent flow:* is an extreme kind of unsteady flow and occurs when there are sharp obstacles or bends in the path of a fast-moving fluid.
- In turbulent flow, the velocity at a point changes erratically from moment to moment, both in magnitude and in direction.

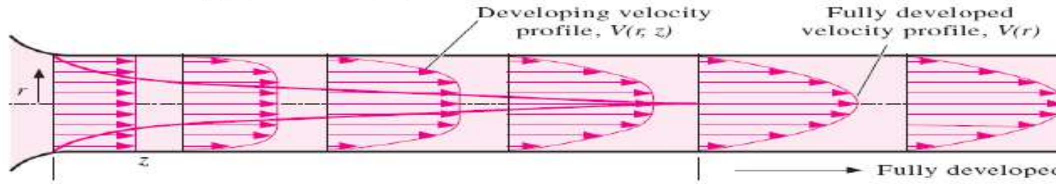
- Because the motion of a real fluid is complex and not yet fully understood, we make some simplifying assumptions in our approach.
- Many features of real fluids in motion can be understood by considering the behavior of an ideal fluid.

Properties of an ideal fluid

In our model of an *ideal fluid*, we make four assumptions:

1. Non-viscous fluid. In a non-viscous fluid, internal friction is neglected. An object moving through the fluid experiences no viscous force.
2. Steady flow. In Steady flow, we assume that the velocity of the fluid at each point remains constant in time.
3. Incompressible fluid. The density of an Incompressible fluid is assumed to remain constant in time.
4. Irrotational flow. Fluid flow is irrotational if there is no angular momentum of the fluid about any point.

- A flow field is best characterized by its velocity distribution.
- A flow is said to be one-, two-, or three-dimensional if the flow velocity varies in one, two, or three dimensions, respectively.



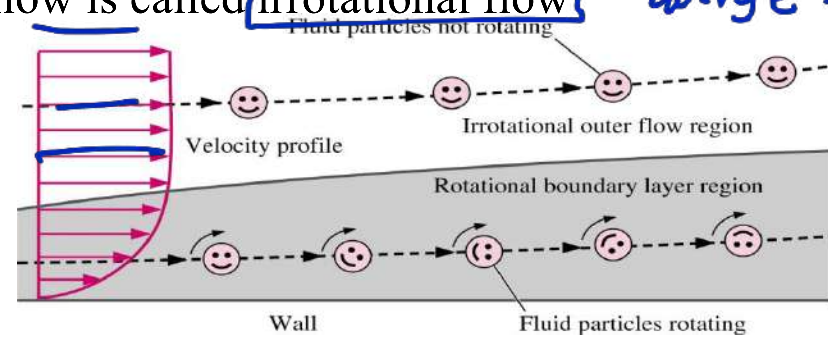
- However, the variation of velocity in certain directions can be small relative to the variation in other directions and can be ignored.

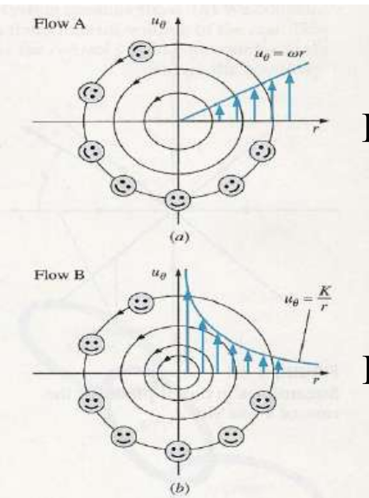
The development of the velocity profile in a circular pipe. V = V(r, z) and thus the flow is two-dimensional in the entrance region, and becomes one-dimensional downstream when the velocity profile fully develops and remains unchanged in the flow direction, V = V(r).

Rotational and irrotational flows

The rotational is the average value of rotation of two $\perp^r$ lines in the flow.

(i) If this average = 0 then there is no rotation and the flow is called irrotational flow averge = 0





Streamlines and velocity profiles for (a) flow A, solid-body rotation and (b) flow B, a line vortex. Flow A is rotational, but flow B is irrotational everywhere except at the origin.

Rotational flow

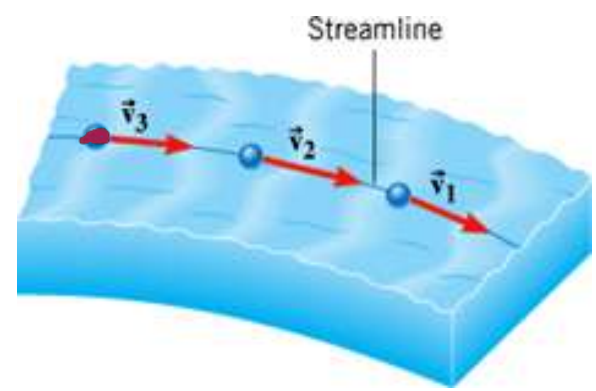
Irrotational flow

Streamline(s): خط انسيابي

- When the flow is steady, ***streamlines*** are often used to represent the trajectories of the fluid particles.
- A streamline is a line drawn in the fluid such that a tangent to the streamline at any point is parallel to the fluid velocity at that point.
- In fact, steady flow is often called ***streamline flow***.

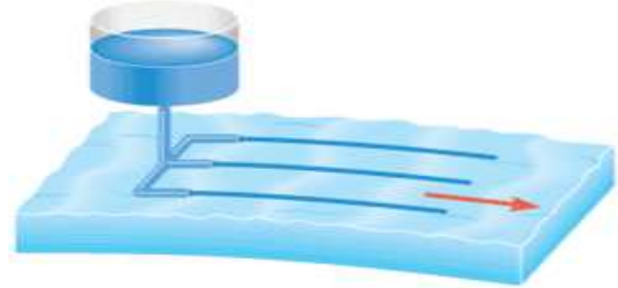
Figure shows the velocity vectors at three points along a streamline.

The fluid velocity can vary (in both magnitude and direction) from point to point along a streamline, but at any given point, the velocity is constant in time, as required by the condition of steady flow.



At any point along a streamline, the velocity vector of the fluid particle at that point is tangent to the streamline.

Streamlines

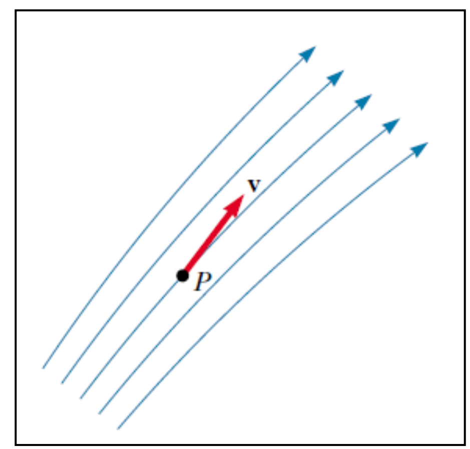


Streamlines:

- Are imaginary lines drawn to show the flow of fluid.
- They are **always parallel to the direction of flow**. Fluid does not flow across streamlines

- ☐ This is diagram represents a set of streamlines (blue lines).
- ☐ A particle at P follows one of these streamlines, and its velocity (v) is tangent to the streamline at each point along its path.
- ☐ No two streamlines can cross each other.

لا يتقاطع خطان من خطوط الجريان



The path taken by a fluid particle under steady flow is called a streamline.

Streamline:

A Streamline is a curve that is everywhere tangent to it at any instant represents the instantaneous local velocity vector.

$$\tan\theta = \frac{dy}{dx} = \frac{v}{u}$$

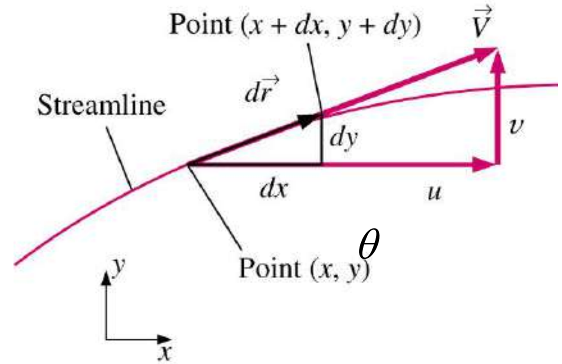
$$\frac{u}{dx} = \frac{v}{dy}$$

$$in - general - for\, 3 - D$$

$$\boxed{\frac{u}{dx} = \frac{v}{dy} = \frac{w}{dz}}$$

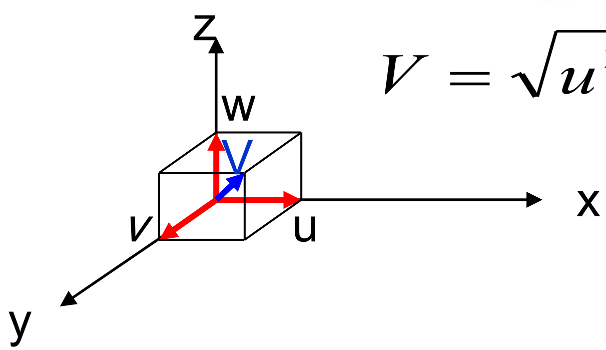
$$\tan\theta = \frac{dy}{dx} = \frac{v}{u}$$

$$\frac{u}{dx} = \frac{v}{dy} = \frac{w}{dz}$$



Stream line equation

معادلة خط جريان:



$$V = \sqrt{u^2 + v^2 + w^2}$$

Where :
u velocity component in -X- direction
v velocity component in-Y- direction
w velocity component in -Z- direction

velocity vector can written as: $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$

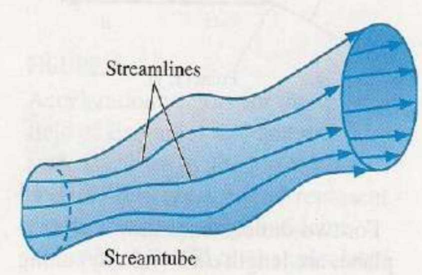
Where : i, j, k are the unit vectors in positive x, y, z direction

انبوب الجريان

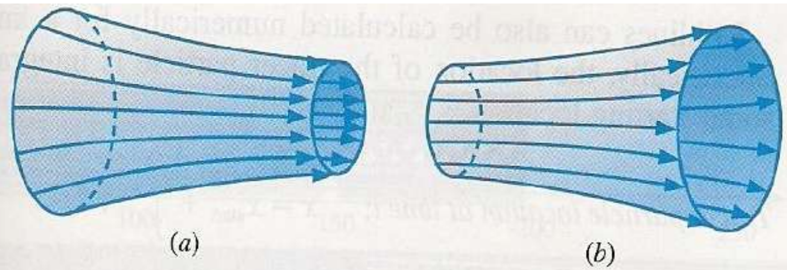
Streamtube:

- Is a bundle of streamlines حزمة من خطوط الجريان
- fluid within a streamtube remain constant and cannot cross the boundary of the streamtube.

  (mass in = mass out)



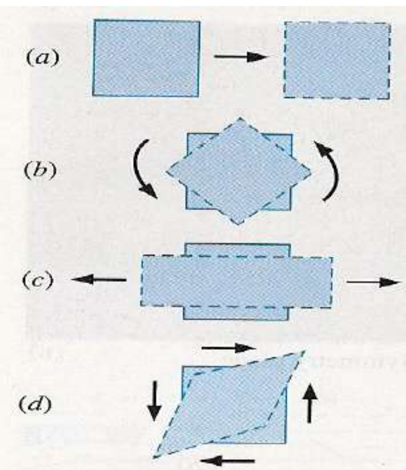
A *streamtube* consists of a bundle of individual streamlines.



In an incompressible flow field, a streamtube (*a*) decreases in diameter as the flow accelerates or converges and (*b*) increases in diameter as the flow decelerates or diverges.

انواع الحركة تشوه

Types of motion or deformation of fluid element

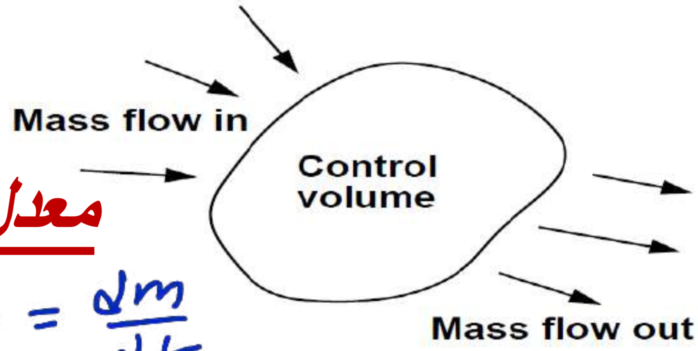


Fundamental types of fluid element motion or deformation: (*a*) translation, (*b*) rotation, (*c*) linear strain, and (*d*) shear strain.

- (a) Linear translation انتقالي
- (b) Rotational translation دوراني
- (c) Linear deformation خطي
- (d) Angular deformation زاوي

Continuity

- This principle of conservation of mass saya matter cannot be created or destroyed.
- This applied in fluids to fixed volumes, known as control volumes (or surfaces)



Mass flow rate: معدل التدفق الكتلي

$$\dot{m} = \frac{m}{t} = \frac{dm}{dt}$$

- The mass of fluid per second (e.g., 5 kg/s) that flows through a tube is called the ***mass flow rate***.
- If a fluid enters one end of a pipe at a certain rate (e.g., 5 kilograms per second), then fluid must also leave at the same rate, assuming that there are no places between the entry and exit points to add or remove fluid.

mass flow rate معدل التدفق الكتلي

$$\dot{m} = \frac{dm}{dt} = \frac{m}{t} = \frac{\text{الكتلة}}{\text{الزمن}} \quad kg/s$$

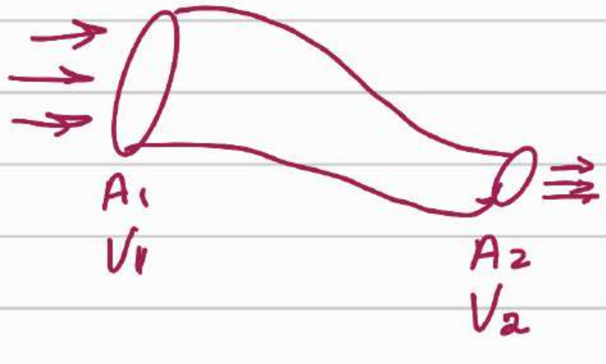
Volume flow rate معدل التدفق الحجمي

$$\text{discharge} = Q = \frac{V}{t} = \frac{\text{الحجم}}{\text{الزمن}}$$

m^3/s

$$Q = A\,v = \text{المساحة} \times \text{السرعة}$$

equation of continuity معادلة الاستمرارية



$$Q_1 = Q_2$$

$$A_1 V_1 = A_2 V_2$$

هذه المعادلة تطبق في حالة المائع الغير قابل للانضغاط

$$A_1 V_1 \rho_1 = A_2 V_2 \rho_2$$

نطبق هذه العلاقة في حالة المائع القابل للانضغاط

$$Q = \frac{\dot{m}}{\rho}$$

علاقة بين Q و $\dot{m}$

Mass flow rate

$$\dot{m} = \frac{dm}{dt} = \frac{\text{mass}}{\text{time taken to accumulate this mass}}$$

A simple example:

An empty bucket weighs 2.0kg. After 7 seconds of collecting water the bucket weighs 8.0kg, then:

8 - 2

$\dot{m} = \frac{6}{7} = 0.857$ kg/s

$$\text{mass flow rate} = \dot{m} = \frac{\text{mass of fluid in bucket}}{\text{time taken to collect the fluid}}$$

$$= \frac{8.0 - 2.0}{7}$$

$$= 0.857\, kg / s$$

Example:

If the mass flow rate is 1.7 kg/s ($\dot{m}$), how long (t) will it take to fill a container with 8kg of fluid? (t=?)

$t = \frac{m}{\dot{m}} = \frac{8}{1.7} = 4.7$ s

$$\text{time} = \frac{\text{mass}}{\text{mass flow rate}}$$

$$= \frac{8}{1.7}$$

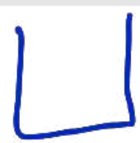
$$= 4.7s$$

Volume flow rate - Discharge.

More commonly we use volume flow rate
Also know as *discharge*.

The symbol normally used for discharge is Q.

$$\text{discharge, } Q = \frac{\text{volume of fluid}}{\text{time}}$$



Example:

If the bucket above fills with 2.0 litres in 25 seconds, what is the discharge?(Q=?)

Solution:

$Q = 2.0 \times 10^{-3}\ m^3/\ 25\ s = 0.00008\ m^3/s$

$$\text{discharge, } Q = \frac{\text{volume of fluid}}{\text{time}}$$

$Q = 0.08\ \ell/s$

Consequently, if the density of the fluid in the above example is 850 kg .m^{-3}, then:

$$\text{discharge, } Q = \frac{\text{volume of fluid}}{\text{time}}$$

$$= \frac{\text{mass of fluid}}{\text{density} \times \text{time}}$$

$$= \frac{\text{mass flow rate}}{\text{density}}$$

$$= \frac{0.857}{850}$$

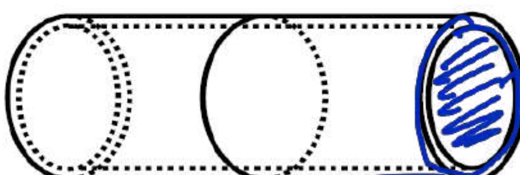
$$= 0.001008\ m^3/s \quad (m^3 s^{-1})$$

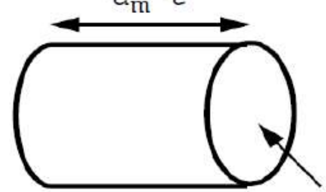
$$= 1.008 \times 10^{-3} m^3/s$$

$$= 1.008 l/s$$

Discharge and mean velocity

If we know the discharge and the diameter of a pipe, we can deduce the *mean* velocity





Cross sectional area of pipe is *A*

Mean velocity is u_m.

In time *t*, a cylinder of fluid will pass point X with a volume $A \times u_m \times t$.

The discharge will thus be

$$Q = \frac{\text{volume}}{\text{time}} = \frac{A \times u_m \times t}{t}$$

$$Q = Au_m = AV$$

u_m متوسط السرعة

V

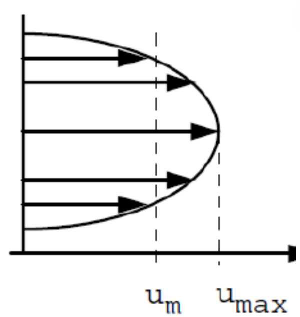
$u = \frac{Q}{A} = \frac{2.4 \times 10^{-3}}{1.2 \times 10^{-3}}$

2m/s

Example:

If A =1.2 × 10^{-3} m^2 and discharge, Q is 2.4 l/s, what is the mean velocity? (u_m=?)

$$u_m = \frac{Q}{A} = \frac{2.4 \times 10^{-3}}{1.2 \times 10^{-3}} = 2.0\, m/s$$



Note how we have called this the *mean* velocity.

This is because the velocity in the pipe is not constant across the cross section.

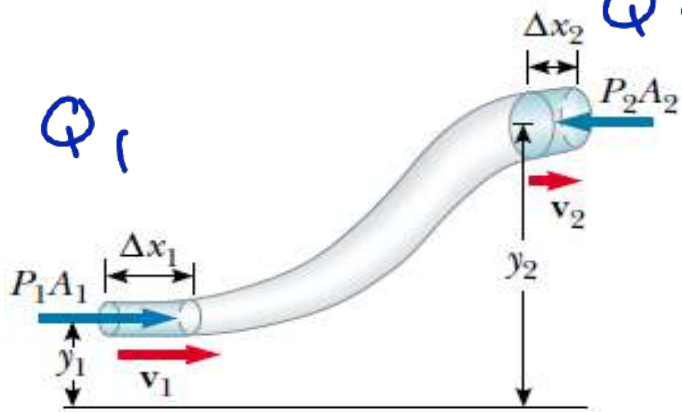
The equation of continuity

معادلة الاستمرارية

A fluid moving with streamline flow through a pipe of varying cross-sectional area. the volume of fluid flowing through A_1 in a time interval Δt must equal the volume flowing through A_2 in the same time interval. **(Fluid is incompressible)**

المائع الغير قابل للانضغاط

$$Q_1 = V_1/t = A_1 L_1/t = A_1 v_1$$



$$Q_2 = V_2/t = A_2 L_2/t = A_2 v_2$$

$$Q_1 = Q_2$$

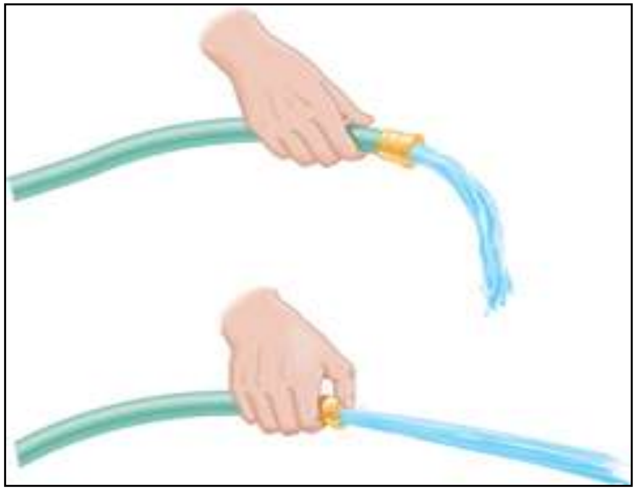
$$A_1 v_1 = A_2 v_2 = \text{Constant}$$

- This expression is called **the equation of continuity**.
- **The product of the area and the fluid speed at all points along the pipe is a constant for an incompressible fluid (continuity principle).**

Ideal fluid flowing through a pipe with changing cross-sectional area.

The volume per unit time ($\Delta v / \Delta t$) passing any point in the pipe must be the same in all parts of the pipe or else we would somehow be creating or destroying fluid. Thus, we get: $A_1 v_1 = A_2 v_2$.
This equation is called the **equation of continuity.**

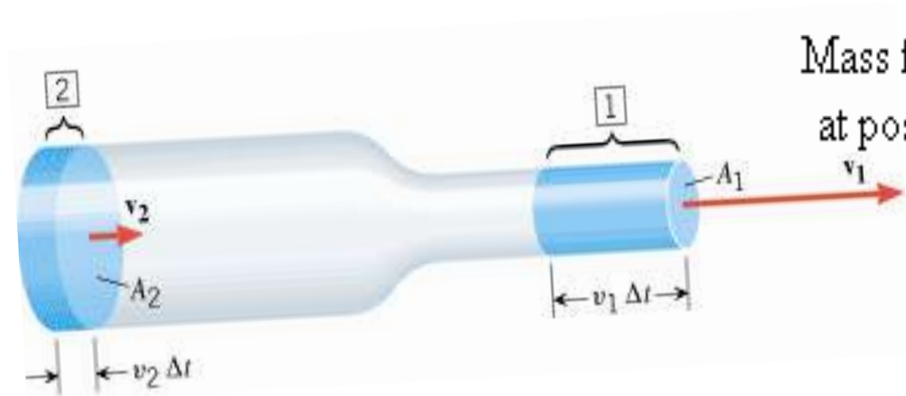
- When the end of a hose is partially closed off, thus reducing its cross-sectional area, the fluid velocity increases.



- This kind of fluid behavior is described by the ***equation of continuity.***

Equation of Continuity

معادلة الاستمرارية



Mass flow rate at position 1 $= \frac{\Delta m_1}{\Delta t} = \rho_1 A_1 v_1$

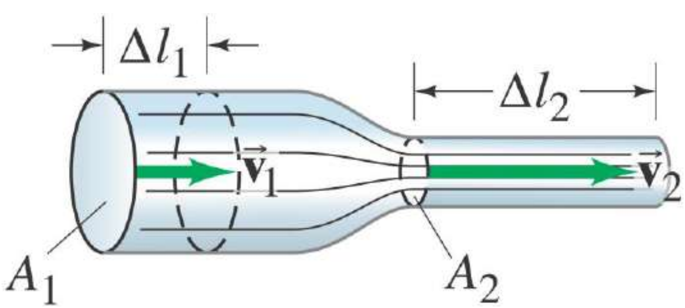
Mass flow rate at position 2 $= \frac{\Delta m_2}{\Delta t} = \rho_2 A_2 v_2$

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

- **If the density <u>doesn't</u> change – typical for liquids – this simplifies to:**

$$A_1 v_1 = A_2 v_2$$

- **Where the pipe is wider, the flow is slower.**



$A_1 > A_2$ but $v_2 > v_1$

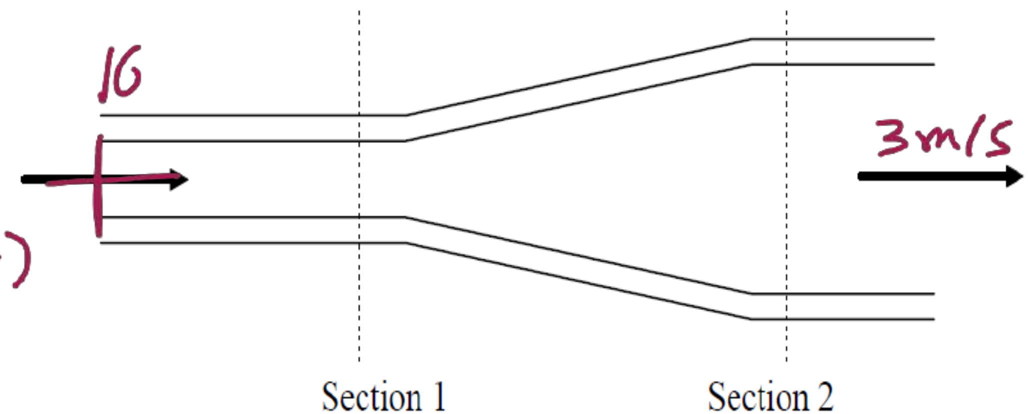
عندما تقل المساحة تزداد السرعة

Mass Flow Rate and the Equation of Continuity

- **We will deal with laminar flow.**
- The **mass flow rate** is the mass that passes a given point per unit time. The flow rates at any two points must be **equal**, as long as no fluid is being added or taken away.
- **This gives us the equation of continuity:**

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

Now try this on a *diffuser*, a pipe which expands or diverges as in the figure below,



$d_1 = 30mm$
$r_1 = 15mm$
$A_1 = \pi(15^2)$

$d_2 = 40mm$
$r_2 = 20mm$
$A_2 = \pi(20)^2$

If d_1=*30mm* and d_2=*40mm* and the velocity u_2=*3.0m/s*.

What is the velocity entering the diffuser?

16 m/s

$$u_1 = \frac{A_2 u_2}{A_1}$$

$$u_1 = \frac{\pi(20)^2 \times 3}{\pi(15)^2}$$

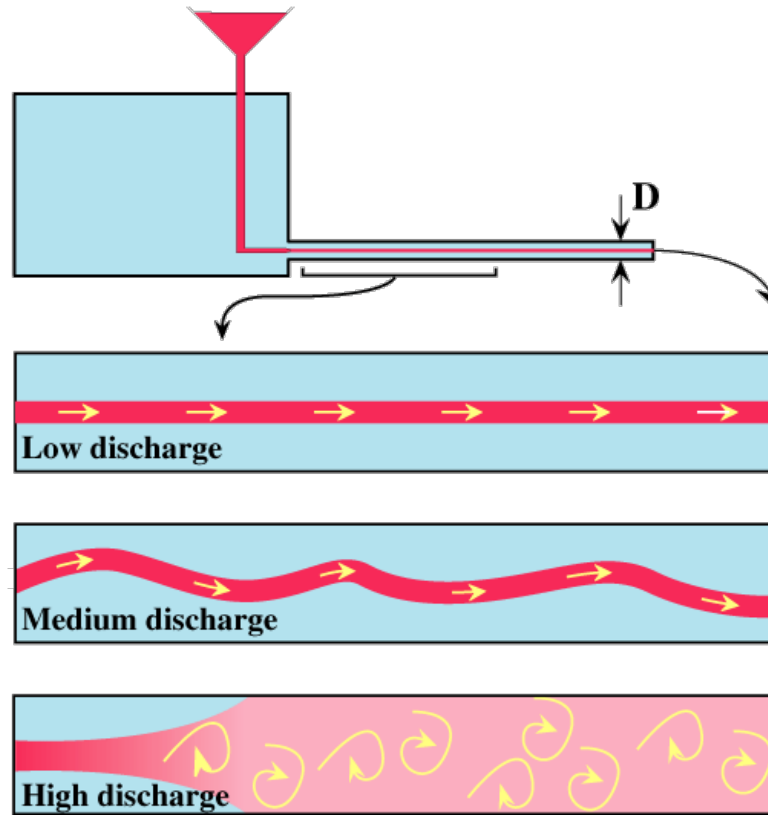
$u_1 = 16$ m/s

Reynold's number
Flow in a pipe or liquid

- ρ: is the density of the fluid
- V: is the mean fluid velocity
- D: is the diameter
- Q: is the volumetric flow rate

$$\mathrm{Re} = \frac{\rho VD}{\mu} = \frac{VD}{\nu} = \frac{QD}{\nu A}$$

- μ: is the dynamic viscosity of the fluid
- ν: is the kinematic velocity of the fluid
- A: is the pipe cross-sectional area.

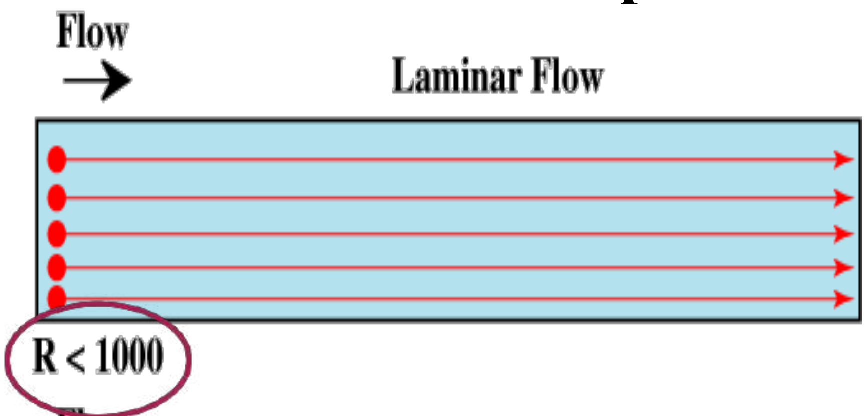


Reynolds' Results

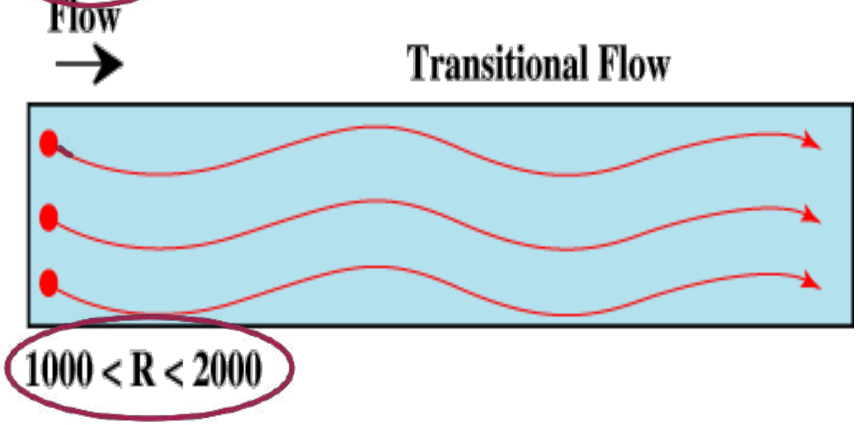
A dimensionless number in fluid mechanics.

The value of R determined the type of flow in the experimental tubes:

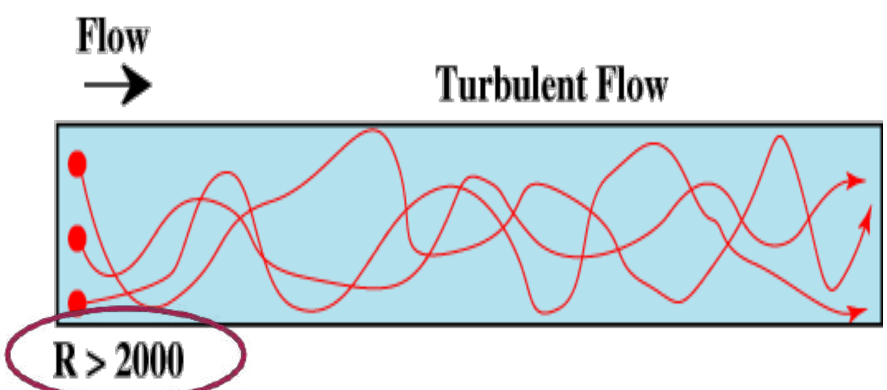
$$R = \frac{UD}{\nu}$$



Laminar Flow: every fluid molecule followed a straight path that was parallel to the boundaries of the tube.



Transitional Flow: every fluid molecule followed wavy but parallel path that was not parallel to the boundaries of the tube.



Turbulent Flow: every fluid molecule followed very complex path that led to a mixing of the dye.

Reynolds found that conditions for each of the flow types depended on:

1. The velocity of the flow (U)
2. The diameter of the tube (D)
3. The density of the fluid (ρ).
4. The fluid's dynamic viscosity (m).

He combined these variables into a dimensionless combination now known as the Flow Reynolds' Number (R) where:

$$\mathbf{R} = \frac{\rho U D}{\mu} \qquad \mathrm{Re} = \frac{\rho u d}{\mu}$$

$$\mathbf{R} = \frac{\rho U D}{\mu} \qquad \mathrm{Re} = \frac{\rho u d}{\mu}$$

Flow Reynolds' number is often expressed in terms of the fluid's *kinematic viscosity* (ν; lower case Greek letter nu), where:

ν

$$\nu = \frac{\mu}{\rho} \qquad \text{(units are m}^2\text{/s)}$$

Rearranging: $\mu = \rho \upsilon$

Substituting into **R**: $\mathbf{R} = \frac{\rho U D}{\rho \upsilon}$ → $\mathbf{R} = \frac{UD}{\nu}$

$$R = \frac{\rho U D}{\mu} \Rightarrow U = \frac{R\mu}{\rho D}$$

$$\rightarrow \frac{2000 \times 1.005 \times 10^{-3}}{998.2 \times 0.1} \simeq 0.02\ \mathrm{m/s}$$

$$\rightarrow \frac{2000 \times 1.005 \times 10^{-3}}{998.2 \times 1} = 0.002\ \mathrm{m/s}$$

Example: Given two pipes, one with a diameter of 10 cm and the other with a diameter of 1 m, at what velocities will the flows in each pipe become turbulent? R = 2000

What is the critical velocity for **R** = 2000?

$$\mathbf{R} = \frac{UD}{\nu} = 2000$$ Solve for U: $$U = \frac{2000\nu}{D}$$

Given: $$\nu = \frac{\mu}{\rho} = \frac{1.005 \times 10^{-3}}{998.2} = 1.007 \times 10^{-6}\,\mathrm{m^2/s}$$ Distilled water at 20°C.

Solve for D = 0.1 m and D = 1.0 m.

For a 0.1 m diameter pipe: $$\boxed{U = \frac{2000\nu}{D}}$$

$$U = \frac{2000 \times 1.007 \times 10^{-6}}{0.1}$$

$$U = 0.02\mathrm{m/s} = 2\mathrm{cm/s}$$

For a 1.0 m diameter pipe:

$$U = \frac{2000 \times 1.007 \times 10^{-6}}{1}$$

$$U = 0.002\mathrm{m/s} = 2\mathrm{mm/s}$$

Frictionless Flow Along Streamlines

- Application of the second Newton's law of motion along streamlines of fluid flow leads to a very famous equation in Fluid Mechanics, i.e. the *Bernoulli equation*.
- There are four assumptions used to derive the equation and these four assumptions must always be remembered to ensure that it is used correctly,

1. The flow is *inviscid* or *frictionless*, i.e. viscous effects are negligible which is valid for low viscosity fluids such as water and air,
2. The flow is *steady*, i.e. the flow pattern is fully developed and does not change with time,

Frictionless Flow along Streamlines

3. The flow is *incompressible*, which is valid for all liquids and low speed gas of Mach 0.3 or below since the change in gas density is less than 5%.

4. The flow considered is *along the same streamline*, as the variation of properties for fluid molecules travelling in the same path can be simulated more accurately through conservation laws of physics.

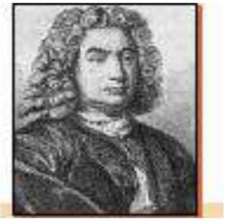
1. الجريان غير لزج أو خالٍ من الاحتكاك، أي أن تأثيرات اللزوجة مهملة. وهو افتراض صالح للموائع منخفضة اللزوجة مثل الماء والهواء.

2. الجريان مستقر أو ثابت، أي أن نمط الجريان متطور بالكامل ولا يتغير مع الزمن.

3. الجريان غير قابل للانضغاط. وهو افتراض صحيح لجميع السوائل وللغازات منخفضة السرعة (عند عدد ماخ أقل من 0.3)، حيث يكون التغير في كثافة الغاز أقل من 5%.

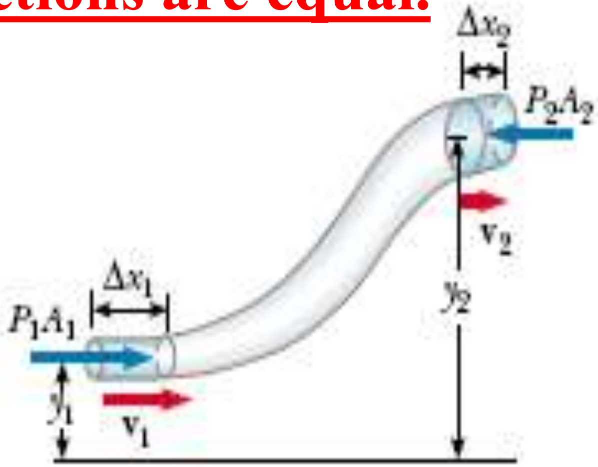
4. الجريان المدروس يكون على طول نفس الخط الانسيابي، حيث يمكن محاكاة تغير خصائص جزيئات المائع التي تسير في نفس المسار بشكل أكثر دقة باستخدام قوانين حفظ الكميات الفيزيائية.

Bernoulli's Equation



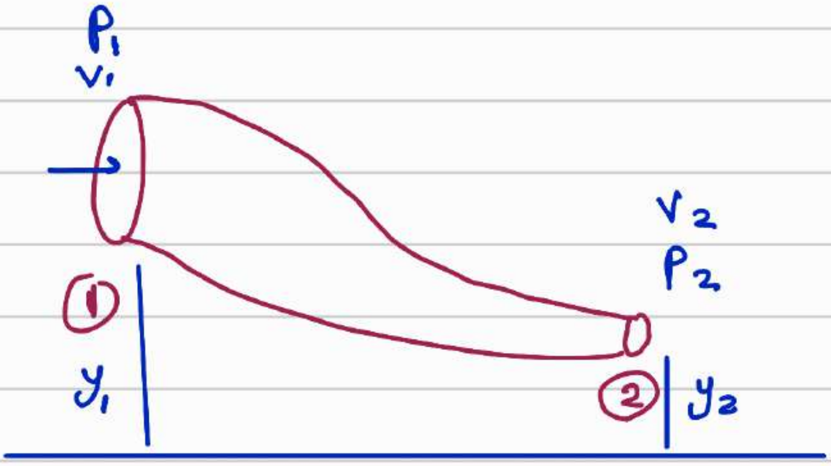
A fluid flowing through a constricted pipe with streamline flow. The fluid in the section of length Δx_1 moves to the section of length Δx_2 .

The volumes of fluid in the two sections are equal.



The speed of water spraying from the end of a hose increases as the size of the opening is decreased with thumb.

معادله برنولي



$$P + \frac{1}{2}\rho v^2 + \rho g y =$$ مقدار ثابت

① ②

$$P_1 + \frac{1}{2}\rho V_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho V_2^2 + \rho g y_2$$

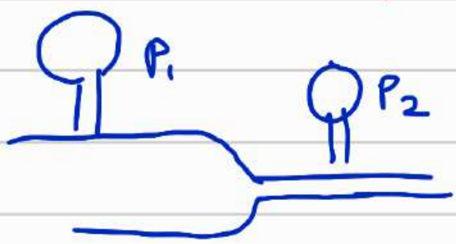
* اذا كان الطرفين عندهم نفس الضغط

$$\frac{1}{2}\rho V_1^2 + \rho g y_1 = \frac{1}{2}\rho V_2^2 + \rho g y_2$$

* اذا كان الطرفين عندهم نفس الارتفاع (انبوب افقي)

$y_1 = y_2$

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$$



مثل

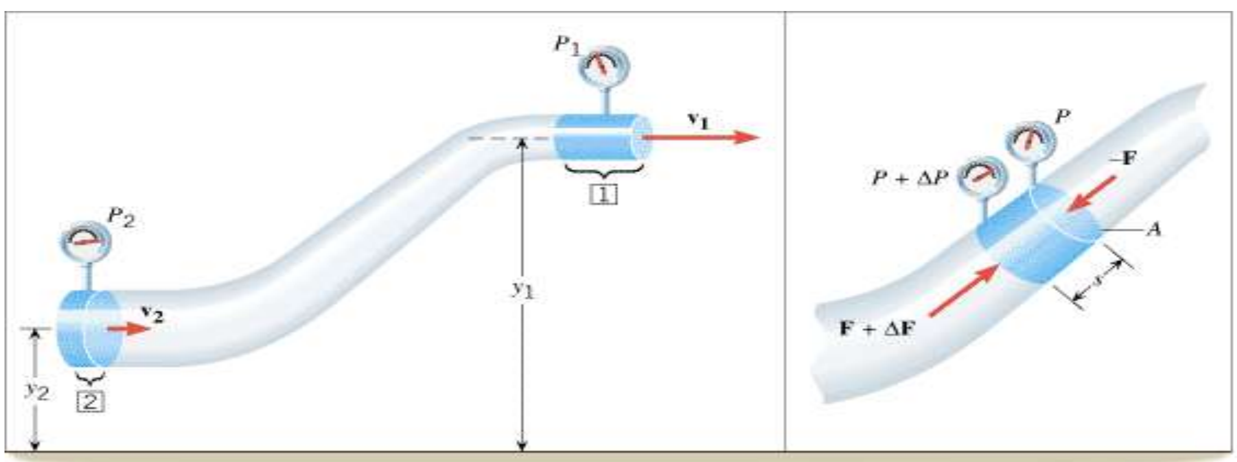
مقياس فنتوري

Bernoulli's Equation:

- The sum of the pressure, kinetic energy per unit volume, and gravitational potential energy per unit volume has the same value at all points along a streamline.
- This result is summarized in Bernoulli's equation:

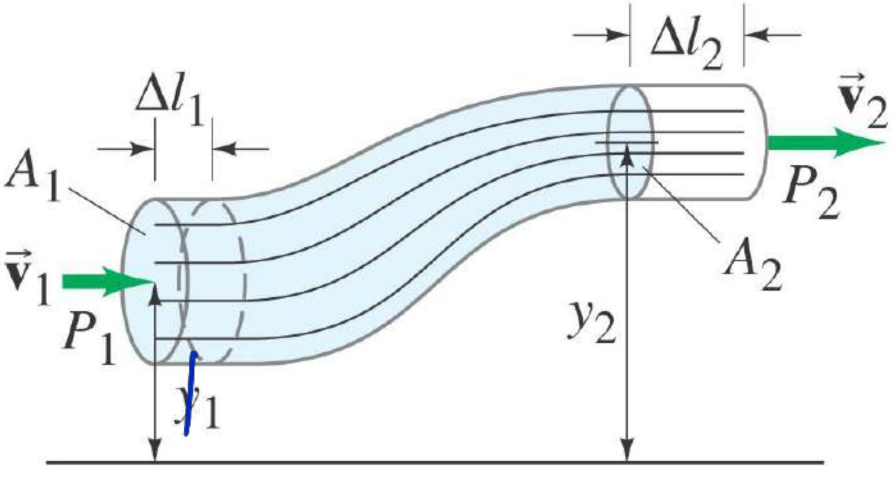
$$P + 1/2\ \rho\ v^2 + \rho\ gy = \text{constant}$$

- For *steady flow*, the speed, pressure, and elevation of an *incompressible* and *nonviscous* fluid are related by an equation discovered by Daniel Bernoulli (1700–1782).



(a) (b)

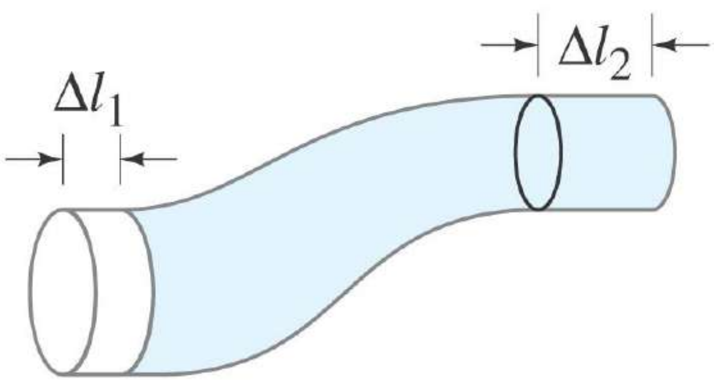
Bernoulli's Equation



(a)

A fluid can also change its **height**. By looking at the work done as it moves, we find:

$$P + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}$$



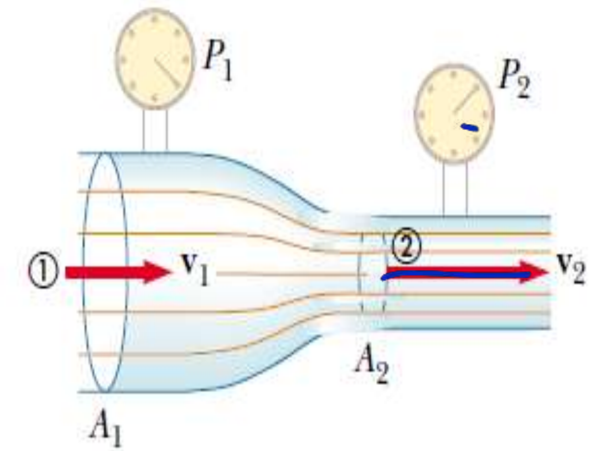
(b)

This is **Bernoulli's** equation. One thing it tells us is that as the **speed** goes up, the **pressure** goes down.

معادلة ١- عندما تزداد السرعه تقل الضغط

انبوب فنتوري

The Venturi tube



Venturi Tube

The horizontal constricted pipe illustrated in figure known as a Venturi tube, can be used to measure the flow speed of an incompressible fluid. Let us determine the flow speed at point 2 if the pressure difference $P_1 - P_2$ is known.

يستخدم لقياس سرعه المائع في انبوب

Solution: Because the pipe is horizontal,

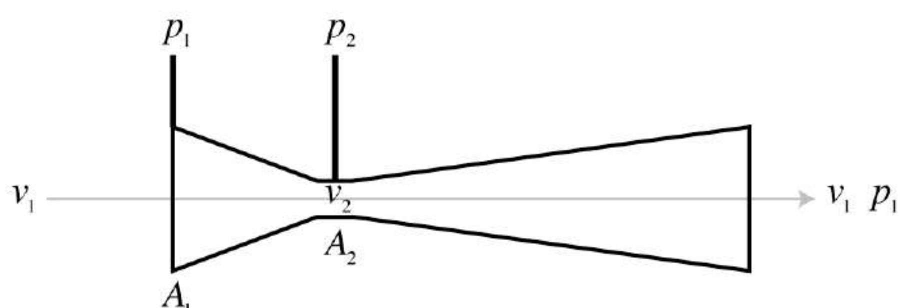
$y_1 = y_2$, and applying Bernoulli's equation:

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

Pressure P_1 is greater Than the pressure P_2 since $v_1 < v_2$. This device can be used to measure the speed of fluid flow.

$P_1 > P_1$

$V_1 < V_2$



A Venturi tube with air flowing through it

Applications of the Bernoulli Theorem

- Flow and the streamline under consideration are shown in Fig. below. Here, using the Bernoulli equation, we can form a relation between point (1) and point (2) as follows:

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho g z_2$$

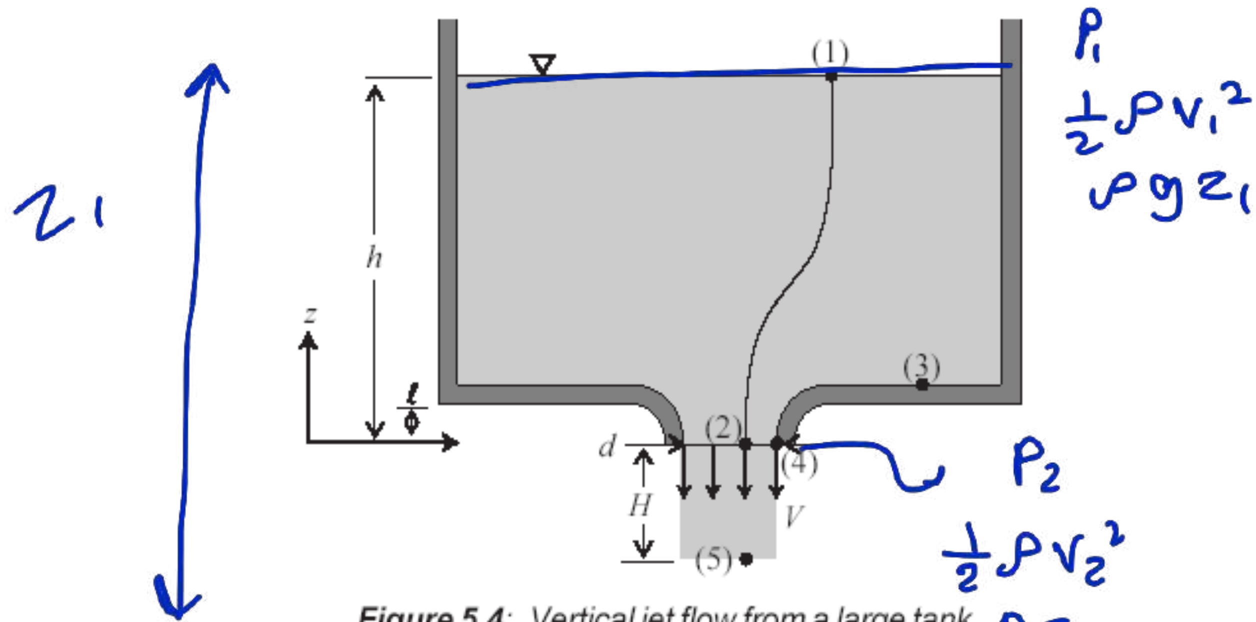
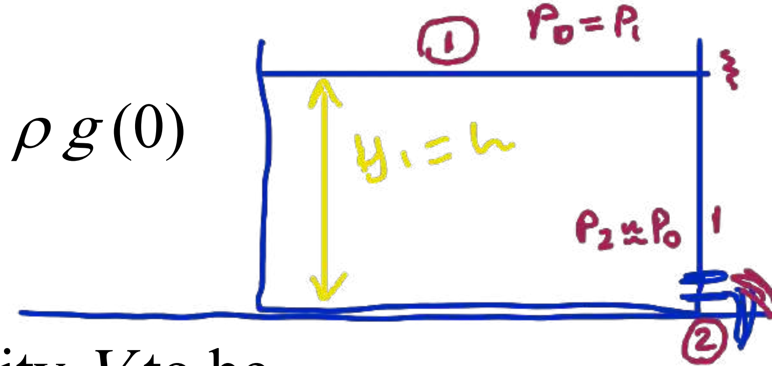


Figure 5.4: *Vertical jet flow from a large tank*

Applications of the Bernoulli Theorem

- At point (1), the pressure is atmospheric ($p_1 = p_0$), or the gauge pressure is zero, and the fluid is almost at rest ($V_1 = 0$). At point (2), the exit pressure is also atmospheric ($p_2 = p_0$), and the fluid moves at a velocity V. By using point (2) as the datum where $z_2 = 0$ and the elevation of point (1) is h, the above relation can be reduced to:



$$p_0 + \tfrac{1}{2}\rho(0)^2 + \rho g h = p_0 + \tfrac{1}{2}\rho V^2 + \rho g(0)$$

$$\rho g h = \tfrac{1}{2}\rho V^2$$

Hence we can formulate the velocity V to be

$$V = \sqrt{2gh}$$

$V_1 \approx 0$ $\quad y_2 \approx 0 \quad P_1 = P_2 \approx P_0$

$$P_1 + \tfrac{1}{2}\rho V_1^2 + \rho g y_1 = P_2 + \tfrac{1}{2}\rho V_2^2 + \rho g y_2$$

$$\rho g h = \tfrac{1}{2}\rho V_2^2$$

$$V = \sqrt{2gh}$$

Irrotational flow

غير دوراني

الجزيئات لاتدور حول مركز كتلتها

- **Irrotational flow** means that no part of the fluid rotates about its ***own center of mass***.

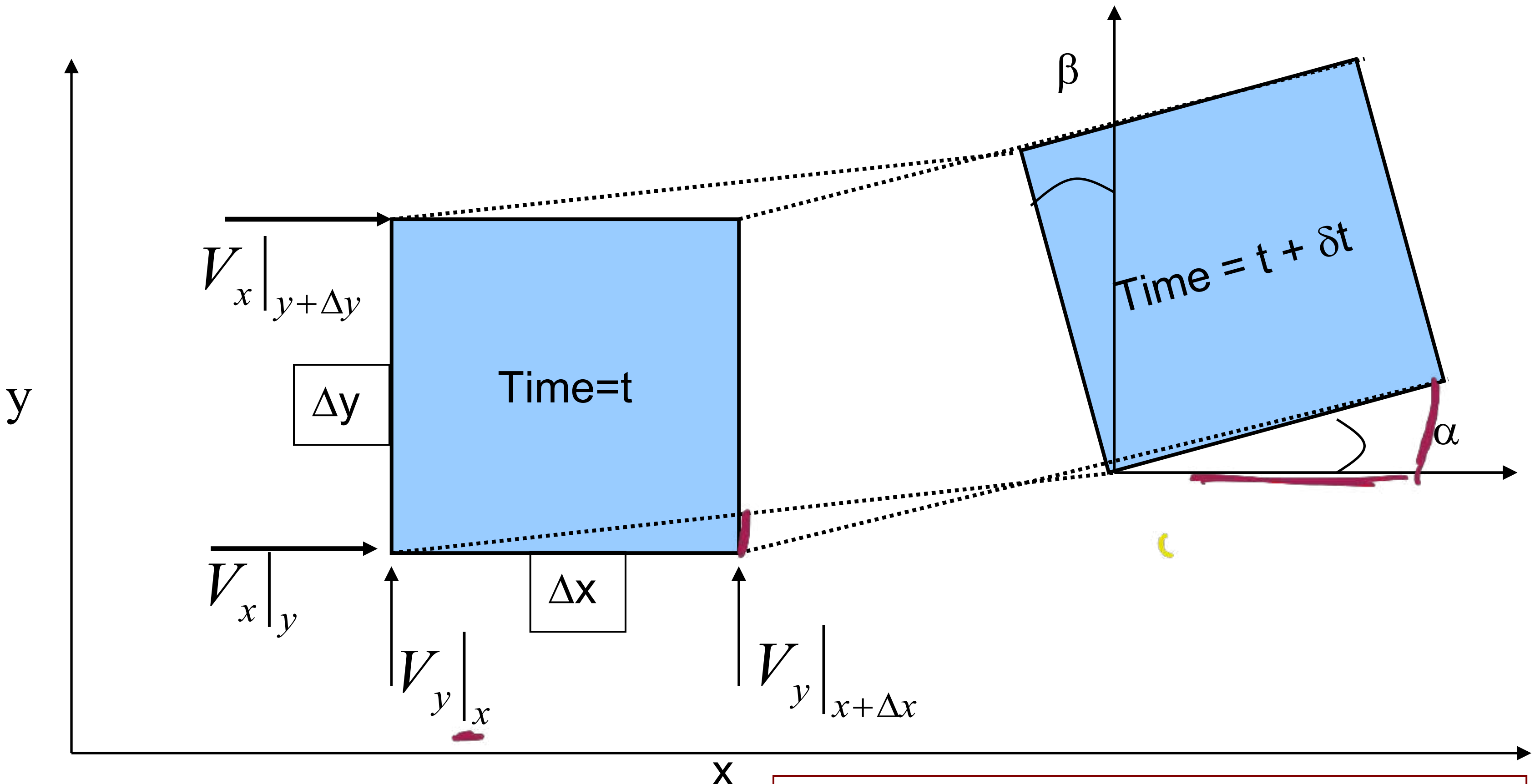
دوراني

- **Rotational motion** of a small part of the fluid would mean that the rotating part of the fluid does have rotational energy, which is not considered in ideal fluids.

يوجد حركة دورانية
يوجد طاقة دورانية
لا يعتبر مائع مثالي

Rotation

Definition of rotation



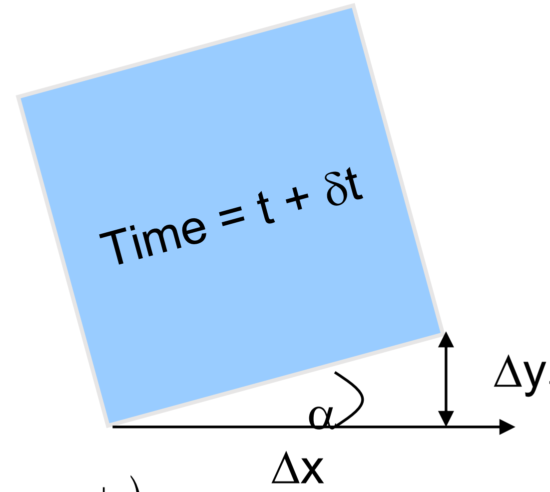
Assume $V_y|_x < V_y|_{x+\Delta x}$

and $V_x|_y > V_x|_{y+\Delta y}$

$$\text{ROTATION} = \omega_z = \frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right)$$

Rotation

To Calculate Rotation $\tan\alpha = \dfrac{\Delta y_1}{\Delta x}$



$$\Delta y_1 = \left(V_y\big|_{x+\Delta x}\Delta t\right) - \left(V_y\big|_{x}\Delta t\right)$$

$$\alpha = \arctan\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_{x}\right)\Delta t}{\Delta x}$$

Similarly

$$\beta = \arctan\frac{-\left(V_x\big|_{y+\Delta y} - V_x\big|_{y}\right)\Delta t}{\Delta y}$$

$$\text{ROTATION} = \omega_z = \frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\lim_{\Delta t\to 0}\left(\frac{(\alpha+\beta)\big|_{t+\Delta t} - (\alpha+\beta)\big|_{t}}{\Delta t}\right)$$

$$= \left(\frac{1}{2}\right)\lim_{\substack{\Delta t\to 0\\ \Delta x\to 0\\ \Delta y\to 0}}\frac{\arctan\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_{x}\right)\Delta t}{\Delta x}\right\}}{\Delta t} - \left(\frac{1}{2}\right)\lim_{\substack{\Delta t\to 0\\ \Delta x\to 0\\ \Delta y\to 0}}\frac{\arctan\left\{\frac{\left(V_x\big|_{y+\Delta y} - V_x\big|_{y}\right)\Delta t}{\Delta y}\right\}}{\Delta t}$$

For very small time and very small element, D_x, D_y and D_t are close to zero

Rotation

For very small q, (i.e.q ~ 0) $\sin\theta \cong \theta$ $\quad \cos\theta \cong 1 \quad \Rightarrow \tan\theta \cong \theta$

$$\therefore \quad \arctan\theta \cong \theta \qquad \therefore \quad \arctan\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}\right\} \cong \frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}$$

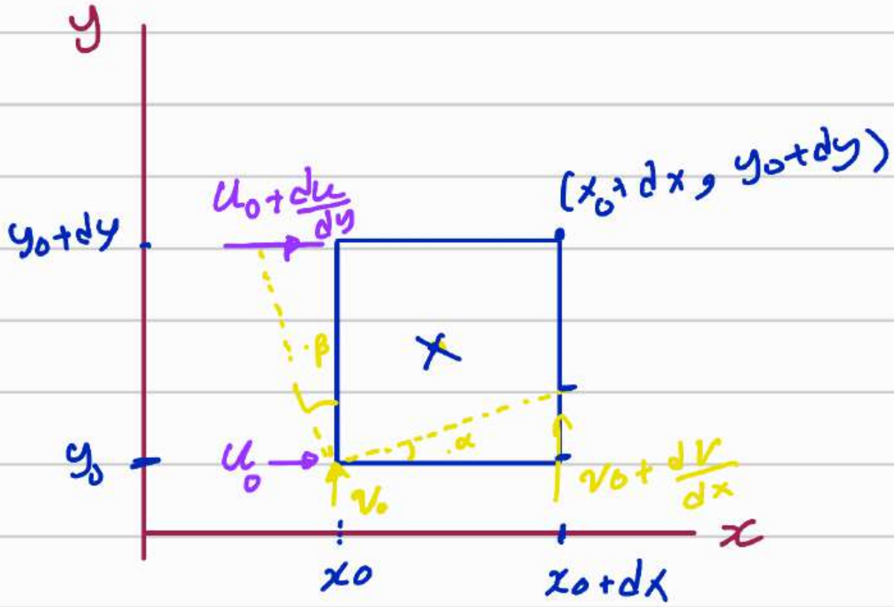
$$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\arctan\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}\right\}}{\Delta t} = \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}\right\}}{\Delta t} \qquad \lim_{\Delta x \to 0} \frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)}{\Delta x} = \frac{\partial V_y}{\partial x}$$

$$\omega_z = \left(\frac{1}{2}\right) \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\arctan\left\{\frac{\left(V_y\big|_{x+\Delta x} - V_y\big|_x\right)\Delta t}{\Delta x}\right\}}{\Delta t} - \left(\frac{1}{2}\right) \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0 \\ \Delta y \to 0}} \frac{\arctan\left\{\frac{\left(V_x\big|_{y+\Delta y} - V_x\big|_y\right)\Delta t}{\Delta y}\right\}}{\Delta t}$$

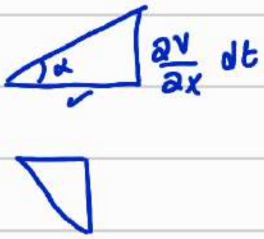
Simplifies to $\omega_z = \left(\frac{1}{2}\right)\left\{\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right\}$ $\quad \omega_z = \left(\frac{1}{2}\right)\left(\nabla \times \vec{V}\right)$

$v = V_y$

$u = V_x$



$$\omega_z = \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{\alpha+\beta}{2}\right)$$



$$d\alpha = \tan\alpha = \frac{\partial v}{\partial x}\,dt$$

$$d\beta = -\tan\beta = -\frac{du}{\partial y}\,dt$$

$u \Rightarrow V_x$

$v \Rightarrow V_y$

$w \Rightarrow V_z$

$$\omega_z = \frac{1}{2\,\cancel{dt}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\cancel{dt}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

($v \to V_y$, $u \to V_x$, $w \to V_z$)

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

الحركة الزاوية

$$\omega = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega = \frac{1}{2}\nabla \times \vec{V}$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Vorticity $\zeta = 2\omega$

$\zeta_x \qquad \zeta_y \qquad \zeta_z$

$$\zeta = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k$$

$$\zeta = \nabla \times \vec{V}$$

اذا كانت الحركة غير دورانية $\omega_x, \omega_y, \omega_z = 0$

تكون الحركة دورانية

ω_x or ω_y or $\omega_z \neq 0$

- **Rotational Flow & Irrotational Flow:**

The rate of rotation can be expressed or equal to the angular velocity vector(ϖ):

$$\varpi = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\bar{i}+\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\bar{j}+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\bar{k}$$

Note:

$$\varpi_x = \frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

$$\varpi_y = \frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$\varpi_z = \frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

The flow is said to be rotational if :

$$\omega_x or\ \omega_y or\ \omega z \neq 0$$

The fluid elements are rotating in space

The flow is said to be irrotational if :

$$\omega_x = \omega_y = \omega_z = 0$$

The fluid elements don't rotating in space

- **Vorticity (ξ):**

Vorticity is a measure of rotation of a fluid particle

Vorticity is twice the angular velocity of a fluid particle

in Cartesian coordinates:

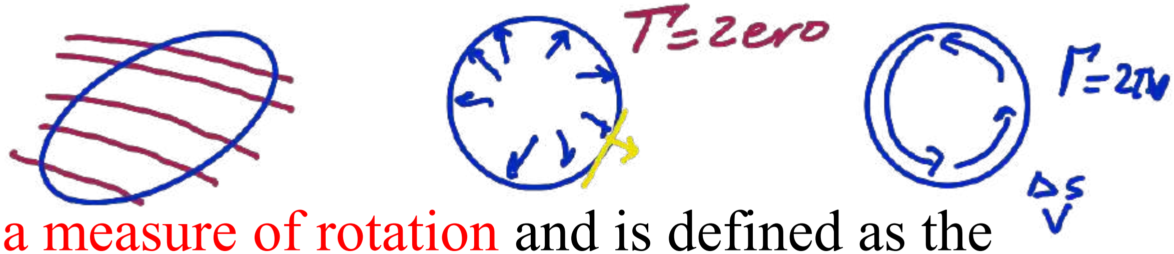
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\vec{i}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\vec{j}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\vec{k}$$

$$\xi_x = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$

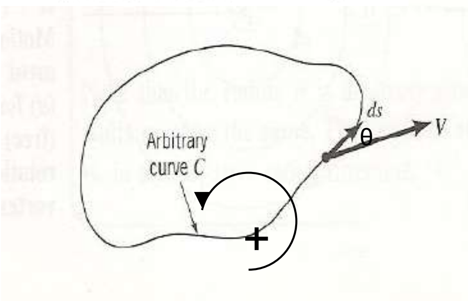
$$\xi_y = \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$\xi_z = \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

- Circulation (Γ):



The circulation (Γ) is a measure of rotation and is defined as the line integral of the tangential component of the velocity taken around a closed curve in the flow field.



$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}$$

The flow is irrotational if ω=0, ξ=0, Γ=0

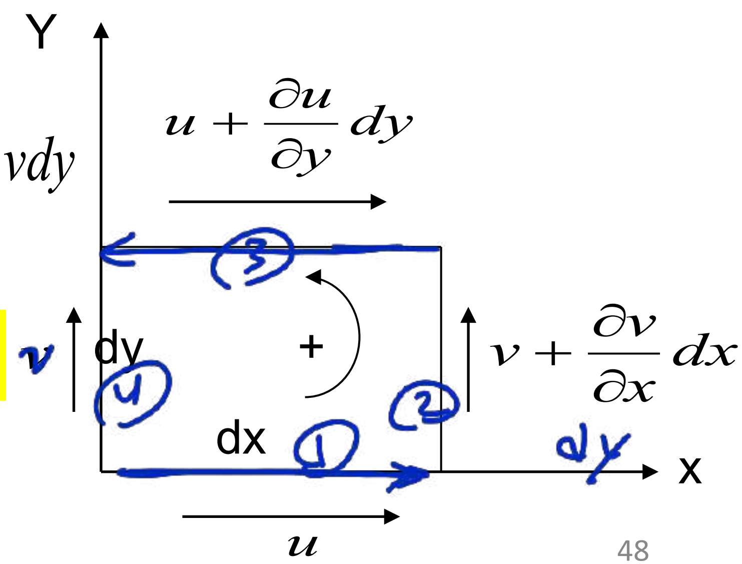
- For 2-D Cartesian Coordinates

$$\Gamma = \int d\Gamma = udx + (v + \frac{\partial v}{\partial x}dx)dy - (u + \frac{\partial u}{\partial y}dy)dx - vdy$$

$$= (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy$$

$$= \zeta_z . area$$

Γ = ξ . area



Γ = circulation

$$\int d\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_3$$

$$= u\,dx + \left(v + \frac{\partial v}{\partial x} dx\right) dy$$

$$- \left(u + \frac{du}{dy} dy\right) dx - v\,dy$$

$$= \underbrace{\left(\frac{\partial v}{\partial x} - \frac{du}{dy}\right)}_{\xi_z} \underbrace{dy\,dx}_{\text{مساحة}}$$

$$\boxed{\Gamma = \xi_z \cdot \text{area}_{xy}}$$

$$a = \frac{dV}{dt} = \underbrace{\frac{\partial V}{\partial t}}_{\text{معتمد عالزمن}} + \underbrace{\frac{\partial V}{\partial x}\overset{u}{\frac{dx}{dt}} + \frac{\partial V}{\partial y}\overset{v}{\frac{dy}{dt}} + \frac{\partial V}{\partial z}\overset{w}{\frac{dz}{dt}}}_{\text{معتمد عالموقع}}$$

$$a = \frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}$$

$$a = \frac{\partial V}{\partial t} + (\nabla \cdot V)$$

➤ Acceleration Field

$a = \dot{V} = \frac{dV}{dt}$

$V = V(x, y, z, t)$

- From Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity.

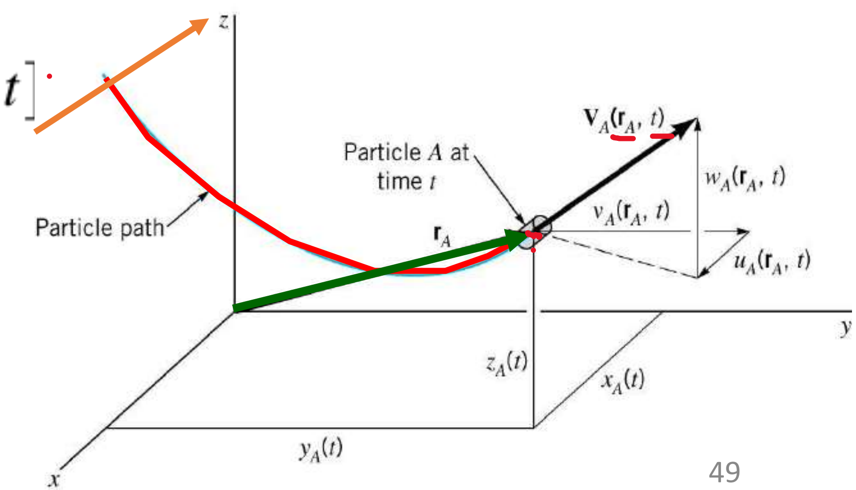
$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

- However, particle velocity at a point is the same as the fluid velocity, Lagrange Frame: $\mathbf{a} = \mathbf{a}(t)$

Eulerian Frame: we describe the acceleration in terms of position and time without following an individual particle. This is analogous to describing the velocity field in terms of space and time.

$$\mathbf{V}_A = \mathbf{V}_A(\mathbf{r}_A, t) = \mathbf{V}_A[x_A(t), y_A(t), z_A(t), t]$$

A fluid particle can accelerate due to a change in velocity in time ("unsteady") or in space (moving to a place with a greater velocity).



Acceleration Field: Material (Substantial) Derivative

$$\mathbf{a}_A(t) = \frac{d\mathbf{V}_A}{dt} = \frac{\partial \mathbf{V}_A}{\partial t} + \frac{\partial \mathbf{V}_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial \mathbf{V}_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial \mathbf{V}_A}{\partial z}\frac{dz_A}{dt}$$

time dependence | **spatial dependence**

We note: $u_A = dx_A/dt$ $\quad v_A = dy_A/dt$ $\quad w_A = dz_A/dt$

Then, substituting: $$\mathbf{a}_A = \frac{\partial \mathbf{V}_A}{\partial t} + u_A\frac{\partial \mathbf{V}_A}{\partial x} + v_A\frac{\partial \mathbf{V}_A}{\partial y} + w_A\frac{\partial \mathbf{V}_A}{\partial z}$$

The above is good for any fluid particle, so we drop "A": $\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$

Writing out these terms in vector components:

x-direction: $a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

y-direction: $a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$

z-direction: $a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$

Fluid flows experience fairly large accelerations or decelerations, especially approaching stagnation points.

Writing these results in "short-hand": $\mathbf{a} = \frac{D\mathbf{V}}{Dt}$

where, $$\frac{D(\ )}{Dt} \equiv \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V}\cdot\nabla)(\ ), \quad \nabla() = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\mathbf{V}\cdot\nabla(\ ) = u\,\partial(\ )/\partial x + v\,\partial(\ )/\partial y + w\,\partial(\ )/\partial z$$

Acceleration Field: Material (Total, Substantial, Substantive) Derivative

يمكن تحقيق علاقة مجال التسارع على مجال الحرارة

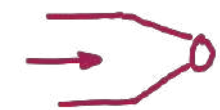
Applied to the Temperature Field in a Flow: $T = T(x, y, z, t)$

$$\mathbf{V} = \mathbf{V}\,(x, y, z, t)$$

The material derivative of any variable is the rate at which that variable changes with time for a given particle (as seen by one moving along with the fluid—Lagrangian description).

$$\frac{dT_A}{dt} = \frac{\partial T_A}{\partial t} + \frac{\partial T_A}{\partial x}\frac{dx_A}{dt} + \frac{\partial T_A}{\partial y}\frac{dy_A}{dt} + \frac{\partial T_A}{\partial z}\frac{dz_A}{dt}$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \boxed{\frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T}$$



$\frac{\partial v}{\partial x} = \checkmark$

$\frac{\partial v}{\partial y} = 0$

$\frac{\partial v}{\partial z} = 0$

$\frac{\partial v}{\partial t} = 0$

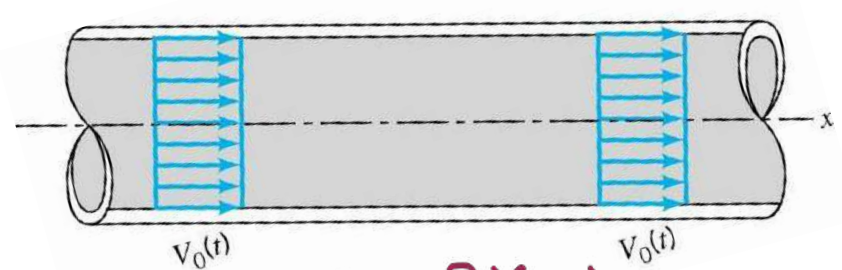
Acceleration Field: Unsteady Effects

Consider flow in a constant diameter pipe, where the flow is assumed to be spatially uniform: $\mathbf{V} = V_0(t)\,\hat{\mathbf{i}}$

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

$\frac{\partial V_0}{\partial t}\hat{\mathbf{i}}$ 0 0 0 0 0

$\Rightarrow \frac{\partial V_0}{\partial t}\hat{\mathbf{i}}$



$a = \frac{\partial v}{\partial t}\, i$

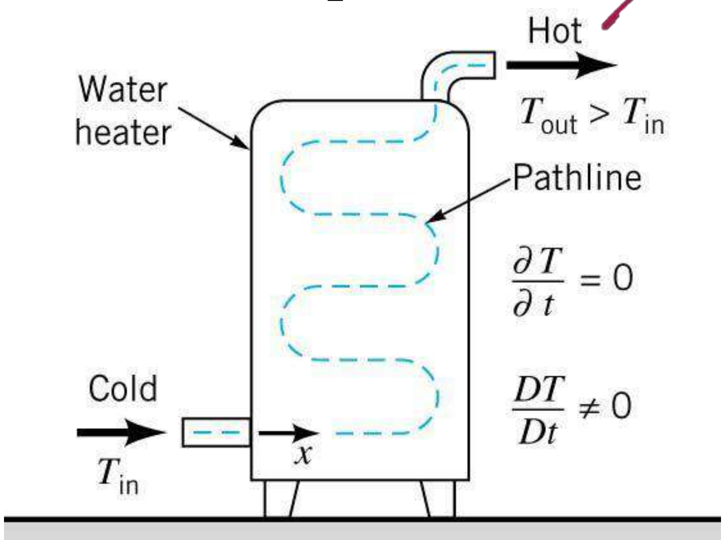
Acceleration Field: Convective Effects

The portion of the material derivative represented by the spatial derivatives is termed the convective term or convective acceleration:

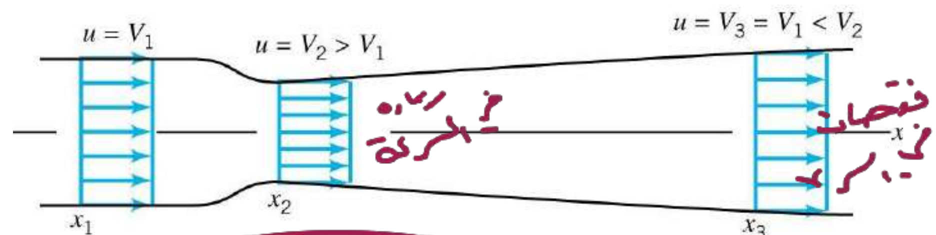
- It represents the fact the flow property associated with a fluid particle may vary due to the motion of the particle from one point in space to another. $(\mathbf{V} \cdot \nabla)\mathbf{V}$

- Convective effects may exist whether the flow is steady or unsteady.

Example 1:



Example 2:



$$a_x = u\, \partial u/\partial x$$

$\partial u/\partial x > 0$ $\quad$ $\partial u/\partial x < 0$

$a_x > 0$ $\quad$ $a_x < 0$

Acceleration = Deacceleration

$$\vec{V} = u(x,y,z,t)\vec{i} + v(x,y,z,t)\vec{j} + w(x,y,z,t)\vec{k}$$

Mathematically the total derivative equals the sum of the partial derivatives

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt$$

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

$$a_x = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Convective component

Local component

Similarly :

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

$$a = \sqrt{a^2{}_x + a^2{}_y + a^2{}_z}$$

Example 2

Consider a hypothetical fluid velocity vector field given by:

$$\boldsymbol{V} = Ct(x^2 - y^2)\,\mathbf{i} - 2Cxyt\,\mathbf{j} + 3y\,\mathbf{k}$$

(i) Is the flow field steady or unsteady?

(ii) Obtain an expression for the acceleration vector $\boldsymbol{a}$.

(iii) Evaluate the acceleration vector $\boldsymbol{a}$ at $(x, y, z, t) = (1, 1, 1, 1)$

$$4C^2 i + (-2C + 4C^2) j - 6C\, k$$

(i) unsteady — يعتمد على زمن

$$a) \quad a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= C(x^2 - y^2) + Ct(x^2 - y^2)\,2xCt$$

$$(-2Cxyt)(-2Cty) + \cancel{3y}(0)$$

$$a_x = C(x^2 - y^2) + 2C^2t^2x(x^2 - y^2) + 4C^2xy^2t^2$$

$$a_x = 4C^2$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$= -2Cxy + 2C^2yt^2(x^2 - y^2) + 4C^2x^2yt^2$$

$$-2C + 4C^2$$

$$a_z = \cancel{\frac{\partial w}{\partial t}} + \cancel{u\frac{\partial w}{\partial t}} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

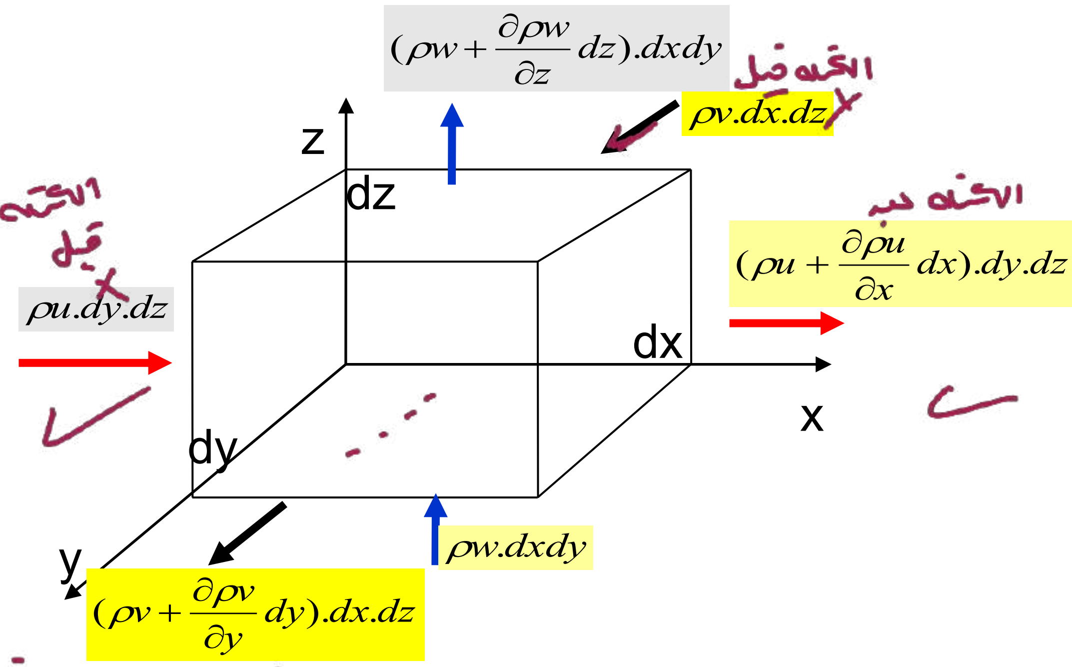
$$a_z = -6Cxyt = -6C$$

Conservation of Mass *(Continuity Equation)*

(Mass can neither be created nor destroyed)

The general equation of continuity for three dimensional steady flow

$m = \rho\, dx\, dy\, dz$



Net mass in x-direction = $\rho u.dy.dz - (\rho u + \frac{\partial \rho u}{\partial x} dx).dy.dz = -\frac{\partial \rho u}{\partial x} dx.dy.dz$

Net mass in y-direction= $\rho v.dx.dz - (\rho v + \frac{\partial \rho v}{\partial y} dy).dx.dz = -\frac{\partial \rho v}{\partial y} dx.dy.dz$

Net mass in z-direction= $\rho w.dx.dy - (\rho w + \frac{\partial \rho w}{\partial z} dz).dx.dy = -\frac{\partial \rho w}{\partial z} dx.dy.dz$

Σ net mass = mass storage rate

$$-\frac{\partial \rho u}{\partial x}dx.dy.dz \quad -\frac{\partial \rho v}{\partial y}dx.dy.dz \quad -\frac{\partial \rho w}{\partial z}dx.dy.dz = \frac{\partial}{\partial t}(\rho dx.dy.dz)$$

$$-\frac{\partial \rho u}{\partial x} \quad -\frac{\partial \rho v}{\partial y} \quad -\frac{\partial \rho w}{\partial z} = \frac{\partial \rho}{\partial t}$$

$$-\frac{\partial \rho}{\partial t} -\frac{\partial \rho u}{\partial x} -\frac{\partial \rho v}{\partial y} -\frac{\partial \rho w}{\partial z} = 0$$

General equation for 3-D , unsteady and compressible fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

Special cases:

1- For steady compressible fluid
2- For incompressible fluid (ρ= constant)

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Note : The above eq$^{n.}$ can be used for steady & unsteady for incompressible fluid

3- For 2-D :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

What is Potential Flow ?

- Basically where:

$$V = \nabla\varphi$$

Velocity — Gradient "del" — Potential function

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

What does this mean?

- If $V = \nabla\varphi$
- Means flow is irrotational. "All potential flow must be irrotational flow".
- φ must satisfy the continuity equation.

Why The Flow Is Irrotational?

$\nabla \times V = 0$ متى تكون الحركة غير دورانية

- For irrotationality the curl of the velocity field is zero.

$V = \nabla\varphi$

$\nabla \times V = \nabla \times \nabla\varphi$

$\nabla \times V = zero$

$$V = \nabla\varphi$$

$$\nabla\times V = \nabla \times \nabla\varphi$$

$$\nabla\times V = 0$$

Curl of gradient is always zero. (from calculus)

For Incompressible Flow

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$\nabla \cdot V = 0$

- If the flow is incompressible we know that:

$$\nabla \cdot V = 0$$

- Remember that,

$$V = \nabla\varphi$$

$$\nabla \cdot \nabla\varphi = 0$$

$$\nabla^2\varphi = 0$$

The Laplace Equation!!!!

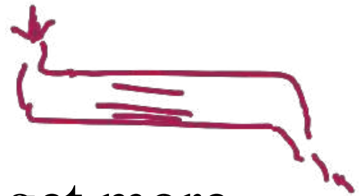
$V = \nabla\varphi$.

$\nabla \cdot \nabla\varphi = \nabla^2\varphi = 0$

- Also this is the continuity equation for potential flow

Why is This Important?

- Superposition Principle
  - If ϕ_1 and ϕ_2 are solutions to the Laplace Equation (*i.e.* Harmonic function) then $\phi_1 + \phi_2$ is also a solution



- Therefore, we can add simple flows together to get more complex flows
  - e.g. Source + sink = doublet
  - Doublet + uniform flow = flow around a cylinder

$$\textit{Flow 3} = \textit{Flow 1} + \textit{Flow 2}$$

$$\phi_{\textit{Flow 3}} = \phi_{\textit{Flow 1}} + \phi_{\textit{Flow 2}}$$

The Problem with Potential Flow

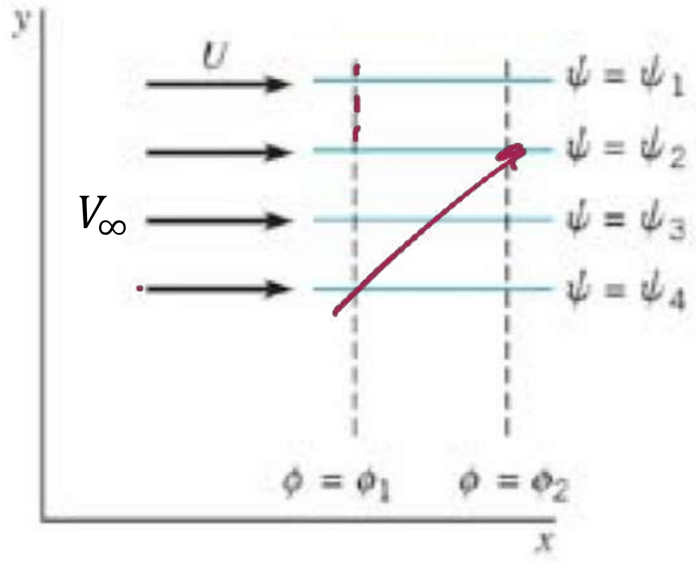
- They don't actually exist
- Don't use near solid bodies
  - These have boundary
- Or anywhere that you expect vorticity
  - E.g. Trailing vorticity of a wing

Potential Flow Model of a Uniform Flow

- Potential function (ϕ), and a Stream function (ψ)
  - If you draw these functions out it will become obvious.
  - In Cartesian coordinates:

$$V_x = u = V_\infty = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$V_y = v = 0$$



Hence:

$$\phi = V_\infty x$$

$$\psi = V_\infty y$$

In Polar-Co-ordinates (r,θ)

- Easy to convert

$$\phi = V_\infty x$$
$$\psi = V_\infty y$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\boxed{\phi = V_\infty r\cos\theta}$$
$$\boxed{\psi = V_\infty r\sin\theta}$$

Velocity Potential

Assume $V_x = \frac{\partial \phi}{\partial x}$

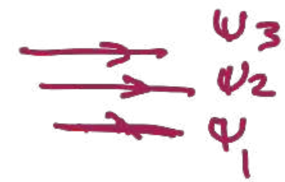
$V_y = \frac{\partial \phi}{\partial y}$

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 0$$

Then $$\frac{\partial V_y}{\partial x} = \frac{\partial V_x}{\partial y} \quad = \frac{\partial^2 \phi}{\partial y \partial x} \quad = \frac{\partial^2 \phi}{\partial x \partial y}$$

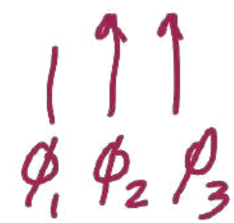
In 3D, similarly it can be shown that $V_z = \frac{\partial \phi}{\partial z}$

f is the velocity potential



Velocity Potential *vs.* Stream Function

	Stream Function (ψ)	**Velocity Potential (ϕ)**
Exists for	only 2D flow	all flows
	viscous or non-viscous flows	Irrotational (i.e. Inviscid or zero viscosity) flow
	Incompressible flow (steady or unsteady)	Incompressible flow (steady or unsteady state)
	compressible flow (steady state only)	compressible flow (steady or unsteady state)



In 2D inviscid flow (incompressible flow OR steady state compressible flow), both functions exist .**What is the relationship between them?**

Stream Function- Physical meaning

Statement: In 2D (viscous or inviscid) flow (incompressible flow steady state compressible flow), y = constant represents the streamline.

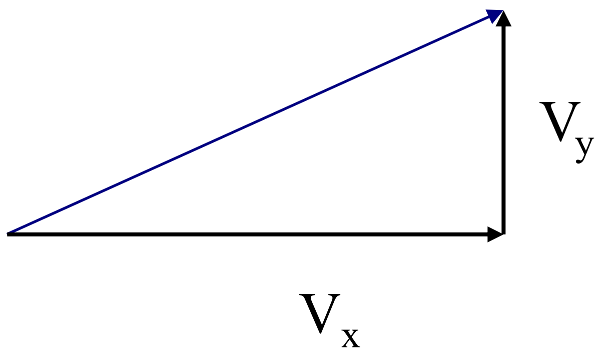
If ψ = constant, then $d\psi = 0$

$$d\psi = \left(\frac{\partial \psi}{\partial x}\right)dx + \left(\frac{\partial \psi}{\partial y}\right)dy$$

$$= \left(-V_y\right)dx + \left(V_x\right)dy$$

$$= 0$$

$$\frac{dy}{dx} = \frac{V_y}{V_x}$$



Stream Function & Velocity Potential

- **Stream Lines**

Consider 2D incompressible flow

Continuity Eq. $\frac{\partial \rho}{\partial t} = 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \frac{\partial}{\partial z}(\rho V_z) = 0$$

$$\frac{\partial}{\partial x}(V_x) + \frac{\partial}{\partial y}(V_y) = 0$$

$$V_y = \int \left(-\frac{\partial V_x}{\partial x} \right) dy$$

V_x and V_y are related

Stream Function & Velocity Potential

Assume $V_x = \dfrac{\partial \psi}{\partial y}$ Then $V_y = \int\left(-\dfrac{\partial V_x}{\partial x}\right)dy = \int\left(-\dfrac{\partial^2 \psi}{\partial x \partial y}\right)dy$

$$= \int\left(-\frac{\partial^2 \psi}{\partial y \partial x}\right)dy = \left(-\frac{\partial \psi}{\partial x}\right)$$

$\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$

Instead of two functions, V_x and V_y, we need to solve for only one function y - Stream Function

Order of differential equation increased by one

$V_x = \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \phi}{\partial x}$

What does Stream Function y mean?

$= \int \dfrac{\partial V_x}{\partial x} dy = \int \dfrac{\partial^2 \psi}{\partial x \partial y}$

Equation for streamlines in 2D are given by

y = constant

ψ ✗

Streamlines may exist in 3D also, but stream function does not

Why? (When we work with velocity potential, we may get a perspective)

In 3D, streamlines follow the equation $\dfrac{dx}{V_x} = \dfrac{dy}{V_y} = \dfrac{dz}{V_z}$

Rotation in terms of Stream Function

To write rotation in terms of stream functions

$$V_x = \frac{\partial \psi}{\partial y} \qquad V_y = \left(-\frac{\partial \psi}{\partial x}\right) \qquad \boxed{\omega_z} = \left(\frac{1}{2}\right)\left\{\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right\} = \left(\frac{1}{2}\right)\left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$= -\left(\frac{1}{2}\right)\nabla^2 \psi$$

That is $\boxed{\nabla^2 \psi + 2\omega_z = 0}$

For irrotational flow ($\boldsymbol{\omega}_z$=0) $\nabla^2 \psi = 0 \qquad \omega_z = \frac{1}{2}\left(\nabla \times \vec{V}\right) = 0$

$$\boxed{\nabla \times \vec{V} = 0} \qquad \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = 0$$

This equation is "similar" to continuity equation V_x and V_y are related.

For the velocity fields given below, determine:

(a) whether the flow field is one-, two-, or three-dimensional, and why.

(b) whether the flow is steady or unsteady, and why.

(The quantities a and b are constants.)

(1) $\vec{V} = [ax^2 e^{-bt}]\hat{i}$

(2) $\vec{V} = ax\hat{i} - by\hat{j}$

$$\vec{V} = ax\hat{i} - by\hat{j}$$

(3) $\vec{V} = ax^2\hat{i} + bx\hat{j} + c\hat{k}$

(4) $\vec{V} = ax^2\hat{i} + bxz\hat{j} + cz\hat{k}$

(5) $\vec{V} = [ae^{-bx}]\hat{i} + bx^2\hat{j}$

(6) $\vec{V} = axy\hat{i} - byzt\hat{j}$

$$\vec{V} = axy\hat{i} - byzt\hat{j}$$

(7) $\vec{V} = a(x^2 + y^2)^{1/2}(1/z^3)\hat{k}$

(8) $\vec{V} = (ax + t)\hat{i} - by^2\hat{j}$

$$\vec{V} = (ax + t)\hat{i} - by2\hat{j}$$

Solution

(1)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(4)	$\vec{V} = \vec{V}(x,z)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(6)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} = \vec{V}(t)$	Unsteady
(7)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady
(8)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady

Problem:

$\frac{v}{u} = \frac{V_y}{V_x} = \frac{dy}{dx}$

A velocity field is given by: $\vec{V} = ax\hat{i} - bty\hat{j}$

Where a = 1 s^{-1} and b = 1 s^{-2} . Find the equation of the streamlines at any time t.

$$\frac{v}{u} = \frac{dy}{dx} = \frac{-bty}{ax}$$

$t=0 \quad y=c$

$t=1 \quad y=\frac{c}{x}$

$t=20 \quad y=cx^{-20}$

$$\int \frac{dy}{y} = \frac{-b}{a} t \int \frac{dx}{x} \Rightarrow \ln y = \frac{-b}{a} t \ln x + c$$

$$\ln y = \ln x^{\frac{-b}{a}t} + c \qquad e^{\ln y} = c\, e^{\ln x^{-\frac{b}{a}t}}$$

$$y = cx^{-\frac{b}{a}t} \qquad y = cx^{-t}$$

Examples

Given : Velocity field, $\vec{V} = Axy\hat{i} + By^2\hat{j}$

A= 1 m^{-1} s^{-1}, B= – 0.5 m^{-1} s^{-1}; coordinates in meter second. Find: an Equation for flow streamlines.

$$\frac{v}{u} = \frac{dy}{dx} = \frac{By^2}{Axy} = \frac{By}{Ax} = \frac{dy}{dx}$$

$$-\frac{y}{2x} = \frac{dy}{dx} \qquad \int 2\frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y^2 = -\ln x + c$$

$$e^{\ln y^2} = e^{\ln x^{-1}} C$$

$$y^2 = x^{-1} C$$

$$\boxed{x y^2 = c}$$

Exercise:

A velocity field is specified as: $\vec{V} = ax^2\hat{i} + bxy\hat{j}$

Where a = 2 $m^{-1}s^{-1}$ and b = − 6 m^{-1} s^{-1}, and the coordinates are measured in meters.

1) Is the flow field one-, two-, or three-dimensional? Why? 2D

2) Calculate the velocity components at the point (2, ½)?

3) Develop an equation for the streamline passing through this point.

7)

$$u = ax^2 = 2(2)^2 = 8 \text{ m/s}$$

$$v = bxy = -6(2)\left(\frac{1}{2}\right) = -6 \text{ m/s}$$

3)

$$\frac{v}{u} = \frac{dy}{dx} = \frac{bxy}{ax^2} = \frac{by}{ax}$$

$$\frac{dy}{dx} = \frac{by}{ax}$$

$$\int \frac{dy}{y} = \frac{b}{a}\int \frac{dx}{x}$$

$$e^{\ln y} = e^{\ln x^{+b/a}} + C$$

$$y = cx^{+b/c}$$

$$y = cx^{-6/2}$$

$$y = cx^{-3}$$

$$y = \frac{c}{x^3}$$