CHAPTER 6

Transcendental Functions

- لوغاريتمات
- الدوال الاسية
- الدوال المثلثية
- الدوال العكسية

6.1 The Natural Logarithm Function (ln)

$\log_2 8 = 3 \qquad 2^3 = 8$

$\ln \quad \cancel{\log}_e m = a \qquad e^a = m \qquad e^x = 4 \qquad \ln 4 = x$

Differential — مشتقة ln

$$\boxed{D_x \ln x = \frac{1}{x}}$$

$$\boxed{D_x \ln \text{دالة} = \frac{\text{مشتقة الدالة}}{\text{الدالة نفسها}} \qquad D_x \ln u = \frac{1}{u} D_x u}$$

$$D_x \ln(3x+1) = \frac{3}{3x+1}$$

$$\frac{d}{dx} \ln \sin x = \frac{\cos x}{\sin x}$$

مثال

find the derevative

a) $\ln 81$

$\frac{d}{dx} \ln 81 = 0$

b) $\ln(x^3+4x+10)$

$\frac{d}{dx} \ln(x^3+4x+10) = \frac{3x^2+4}{x^3+4x+10}$

EXAMPLE 1 Find $D_x \ln \sqrt{x}$. → المشتقة / الدالة

$u = \sqrt{x} = x^{1/2}$ $\quad D_x u = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$D_x \ln u = \frac{D_x u}{u} = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{x}} = \frac{1}{2x}$

EXAMPLE 2 Find $D_x \ln(x^2 - x - 2)$.

$D_x \ln(x^2 - x - 2) = \frac{\text{المشتقة}}{\text{الدالة}}$

$= \frac{2x-1}{x^2-x-2}$

EXAMPLE 3 Show that

$$\boxed{D_x \ln|x| = \frac{1}{x}, \qquad x \neq 0}$$

$D_x \ln|x| = \begin{cases} D_x \ln x = \frac{1}{x} \\ D_x \ln -x = \frac{-1}{-x} = \frac{1}{x} \end{cases}$

دائماً الجواب $\frac{1}{x}$

تكامل اللوغاريتم الطبيعي

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \frac{3}{4x+1}\,dx = 3\int \frac{1}{4x+1}\,dx = \frac{3}{4}\ln|4x+1| + c$$

$$\int \frac{1}{u}\,du = \int \frac{\text{مشتقة}}{\text{الدالة الأصلية}}\,dx = \ln\left(\underset{u}{\text{الدالة}}\right)$$

$$\int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$$

EXAMPLE 4 Find $\int \frac{5}{2x+7}\,dx$.

$$5\int \frac{1}{2x+7}\,dx = \frac{5}{2}\ln|2x+7| + c$$

ملاحظة

$u = 2x+7 \qquad \frac{du}{2} = dx$

نعوض كل x بـ u

$$\int \frac{5}{2x+7}\,dx = \int \frac{5}{u}\,\frac{du}{2} = \frac{5}{2}\int \frac{du}{u}$$

$$\frac{5}{2}\ln u + c = \frac{5}{2}\ln|2x+7| + c$$

EXAMPLE 5 Evaluate $\int_{-1}^{3} \frac{x}{10 - x^2}\, dx$.

نضرب البسط والمقام بـ -2

$$\frac{1}{-2}\int \frac{-2x}{10-x^2}\,dx = -\frac{1}{2}\ln|10-x^2|$$

$u = 10 - x^2 \qquad du = -2x\,dx \qquad dx = \frac{du}{-2x}$

نعوض بدلاله u

$$\int \frac{x}{10-x^2}\,dx = \int \frac{\cancel{x}}{u}\,\frac{du}{-2\cancel{x}} = \frac{1}{-2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|10-x^2|\Big]_{-1}^{3}$$

$$= -\frac{1}{2}\left[\ln|10-3^2| - \ln|10-(-1)^2|\right]$$

$$= -\frac{1}{2}\left[\ln 1 - \ln 9\right] = -\frac{1}{2}\ln 1 + \frac{1}{2}\ln 9$$

EXAMPLE 6 Find $\int \frac{x^2 - x}{x + 1}\, dx$.

لان درجه البسط اكبر من المقام

نقسم قسمه طويله

$$\frac{\text{الباقي}}{\text{المقسوم عليه}} + \text{الناتج}$$

$$\begin{array}{r} x-2 \\ x+1 \overline{)\, x^2 - x} \\ \underline{-x^2 \mp x} \\ -2x \\ \underline{\mp 2x \mp 2} \\ 2 \end{array}$$

$$\int \frac{x^2-x}{x+1}\,dx = \int\left[(x-2) + \frac{2}{x+1}\right]dx$$

$$= \int (x-2)\,d + 2\int \frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2} - 2x + 2\ln|x+1| + C$$

EXAMPLE 7 Find dy/dx if $y = \ln \sqrt[3]{(x-1)/x^2}$, $x > 1$.

احسب المشتقة

$$y = \ln \sqrt[3]{\frac{x-1}{x^2}} \qquad x > 1$$

نستخدم قواعد اللوغاريتمات

$$\ln a^n = n \ln a \qquad \ln ab = \ln a + \ln b$$

$$\ln \frac{a}{b} = \ln a - \ln b$$

$$y = \ln\left(\frac{x-1}{x^2}\right)^{1/3} = \frac{1}{3} \ln \frac{x-1}{x^2}$$

$$y = \frac{1}{3}\left(\ln(x-1) - \ln x^2\right)$$

$$y = \frac{1}{3}\left(\ln(x-1) - 2\ln x\right)$$

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x-1} - 2\,\frac{1}{x}\right]$$

$$= \frac{1}{3}\left[\frac{x}{x(x-1)} - \frac{2(x-1)}{x(x-1)}\right]$$

$$= \frac{1}{3}\left[\frac{x-2x+2}{x(x-1)}\right] = \frac{-x+2}{3x(x-1)}$$

EXAMPLE 8 Differentiate $y = \frac{\sqrt{1-x^2}}{(x+1)^{2/3}}$. $= \frac{(1-x^2)^{1/2}}{(x+1)^{2/3}}$

$$\frac{dy}{dx} = \frac{f}{g} = \frac{gf' - fg'}{g^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(x+1)^{2/3}(1-x^2)^{-1/2}(2x) - \frac{2}{3}(1-x^2)^{1/2}(x+1)^{-1/3}}{(x+1)^{4/3}}$$

⋮

نبسط اكل

هل آخر ندخل ln لطرفي المعادلة

$3y^2 = 4x+5$

$6y\frac{dy}{dx} = 4$

$$\ln y = \ln \frac{\sqrt{1-x^2}}{(x+1)^{2/3}}$$

$$\ln y = \ln(1-x^2)^{1/2} - \ln(x+1)^{2/3}$$

$$\ln y = \frac{1}{2}\ln(1-x^2) - \frac{2}{3}\ln(x+1)$$

نشتق الطرفين

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\,\frac{-2x}{1-x^2} - \frac{2}{3}\,\frac{1}{x+1}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-x}{1-x^2} - \frac{2}{3(x+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-x}{(1+x)(1-x)}^{\times 3} - \frac{2}{3(x+1)}^{(1-x)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-3x-2+2x}{3(1+x)(1-x)} = \frac{-2-x}{3(1-x^2)}$$

$$\frac{dy}{dx} = -y\,\frac{(2+x)}{3(1-x^2)}$$

$$\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-x^2}} =$$

$$\frac{dy}{dx} = \frac{-\sqrt{1-x^2}}{3\,(x+1)^{2/3}}\;\frac{(2+x)}{(1-x^2)}$$

$$\frac{dy}{dx} = \frac{-(x+2)}{3(x+1)^{2/3}(1-x^2)^{1/2}}$$

EXAMPLE 9 Evaluate $\int \tan x\, dx$.

$$\frac{-\sin x}{\cos x}$$

$$-\int -\frac{\sin x}{\cos x}\, dx = -\ln|\cos x| + C$$

$$\int \frac{\sin x}{\cos x}\, dx = \int \frac{\cancel{\sin x}}{u}\,\frac{du}{\cancel{-\sin x}} = -\int \frac{1}{u}\, du = -\ln|\cos x|$$

$\cos x = u \qquad du = -\sin x\, dx \qquad dx = \frac{du}{-\sin x}$

EXAMPLE 10 Evaluate $\int \sec x \csc x\, dx$.

$\sec x \csc x = \tan x + \cot x$

$$\int \tan x\, dx + \int \cot x\, dx \qquad \int \frac{1}{u}\, du$$

$$\int \frac{\sin x}{\cos x}\, dx + \int \frac{\cos x}{\sin x}$$

$$-\ln|\cos x| + \ln|\sin x| + C$$

Problem Set 6.1

In Problems 3–14, find the indicated derivative (see Examples 1 and 2). Assume in each case that x is restricted so that ln is defined.

3. $D_x \ln(x^2 + 3x + \pi)$

5. $D_x \ln(x - 4)^3$

3) $\dfrac{2x + 3}{x^2 + 3x + \pi}$

5) $\dfrac{3(x-4)^2}{(x-4)^3} = \dfrac{3}{(x-4)}$

7. $\dfrac{dy}{dx}$ if $y = 3 \ln x$

7) $y' = 3\,\dfrac{1}{x} = \dfrac{3}{x}$

9. $\dfrac{dz}{dx}$ if $z = x^2 \ln x^2 + (\ln x)^3$ $= 2x^2 \ln x + (\ln x)^3$

9) $2x^2\,\dfrac{1}{x} + 4x \ln x + 3(\ln x)^2\,\dfrac{1}{x}$

$2x + 4x \ln x + \dfrac{3}{x}(\ln x)^2$

13. $f'(81)$ if $f(x) = \ln \sqrt[3]{x}$ $= \ln x^{1/3} = \dfrac{1}{3} \ln x$

$f'(x) = \dfrac{1}{3}\,\dfrac{1}{x} = \dfrac{1}{3x}$

$f'(81) = \dfrac{1}{3 \cdot 81} = \dfrac{1}{243}$

In Problems 31–34, find dy/dx by logarithmic differentiation (see Example 8).

31. $y = \dfrac{x+11}{\sqrt{x^3-4}}$

$$\ln y = \ln \frac{x+11}{\sqrt{x^3-4}}$$

$$= \ln(x+11) - \ln(x^3-4)^{1/2}$$

$$\ln y = \ln(x+11) - \frac{1}{2}\ln(x^3-4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{1}{2}\frac{3x^2}{x^3-4}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{3x^2}{2(x^3-4)}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4}\right]$$

$$\frac{dy}{dx} = \frac{x+11}{\sqrt{x^3+4}}\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4)}\right]$$

33. $y = \dfrac{\sqrt{x+13}}{(x-4)\sqrt[3]{2x+1}}$

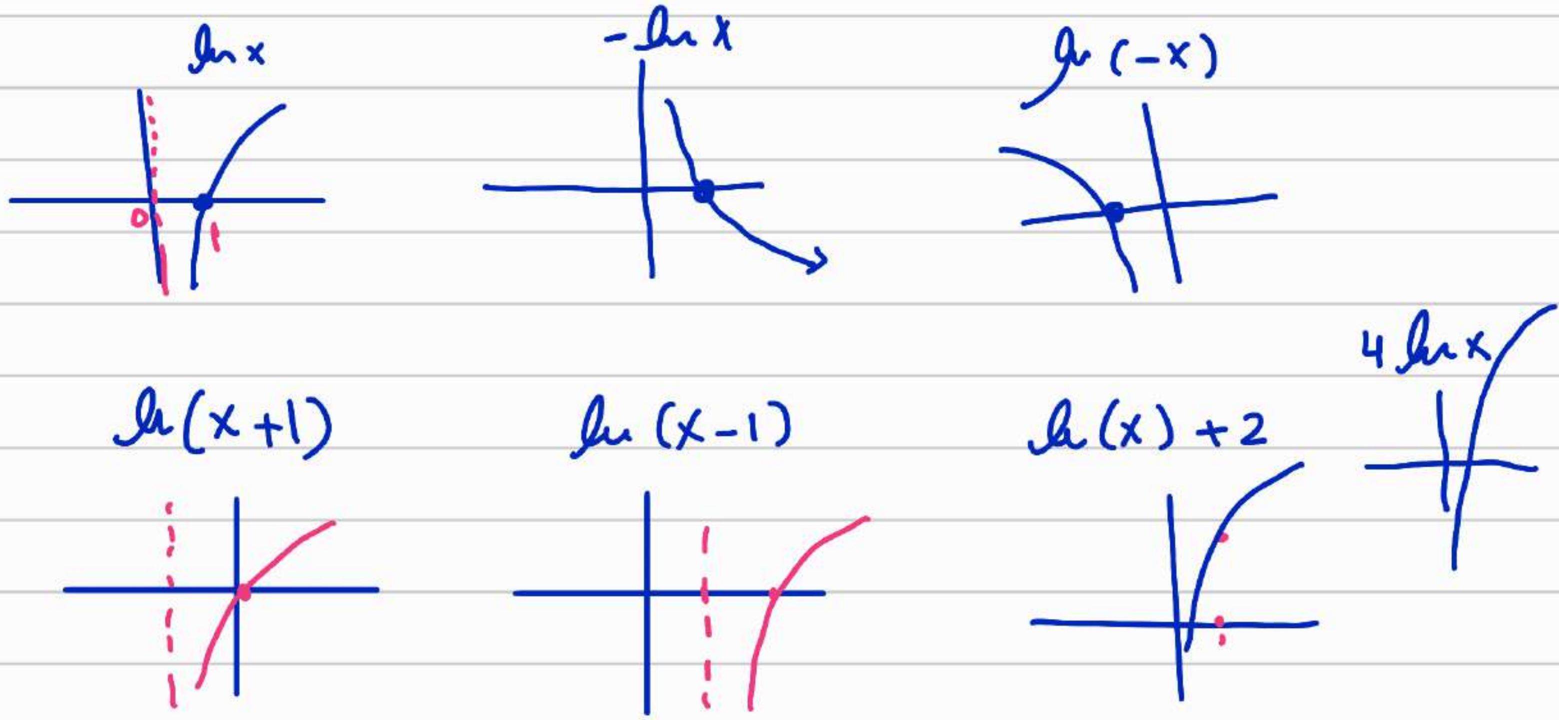
$$\ln y = \ln (x+3)^{1/2} - \ln(x-4) - \ln(2x+1)^{1/3}$$

$$\ln y = \frac{1}{2} \ln(x+13) - \ln(x-4) - \frac{1}{3} \ln(2x+1)$$

$$\frac{1}{y} y' = \frac{1}{2} \frac{1}{x+13} - \frac{1}{x-1} - \frac{1}{3} \frac{2}{2x+1}$$

$$y' = \frac{1}{2(x+13)} - \frac{1}{(x-1)} - \frac{2}{3(2x+1)} \cdot y$$

$$y' = \left[\frac{1}{2(x+13)} - \frac{1}{x-1} - \frac{2}{3(2x+1)}\right] \frac{\sqrt{x+13}}{(x-4)\sqrt[3]{2x+1}}$$

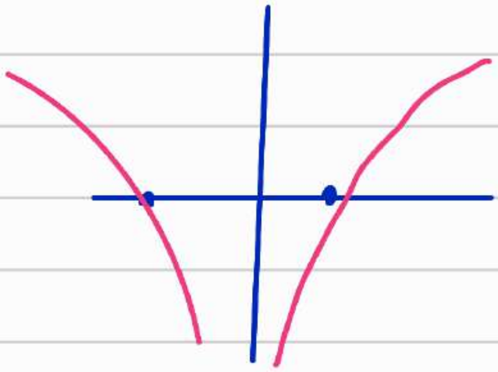


In Problems 35–38, make use of the known graph of $y = \ln x$ to sketch the graphs of the equations.

35. $y = \ln|x|$ **36.** $y = \ln\sqrt{x}$

37. $y = \ln\left(\dfrac{1}{x}\right)$ **38.** $y = \ln(x - 2)$

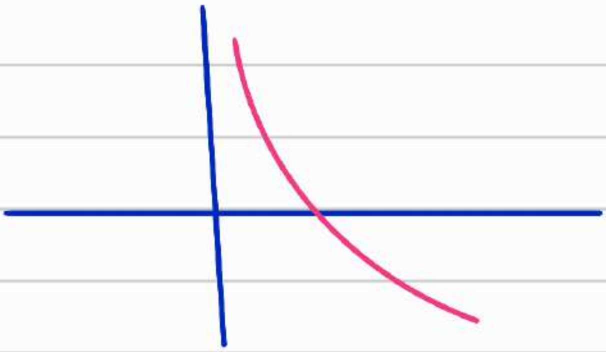
35) $\ln|x| \begin{cases} \ln x \\ \ln -x \end{cases}$



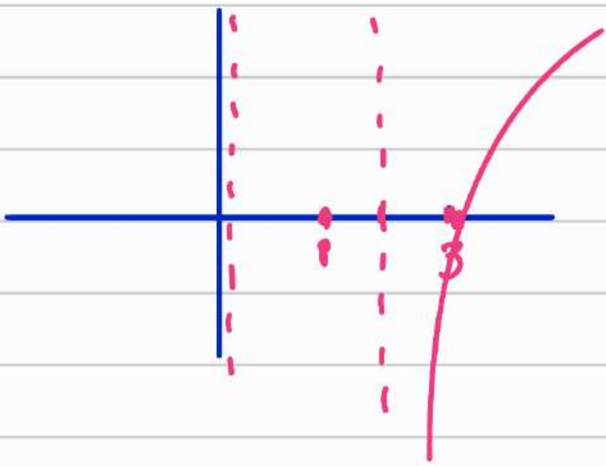
73) $y = \ln\left(\dfrac{1}{x}\right)$

$y = \ln x^{-1}$

$= -\ln x$



38)



Problem Set 6.1

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4. $D_x \ln(3x^3 + 2x)$

5. $D_x \ln(x - 4)^3$

6. $D_x \ln \sqrt{3x - 2}$

7. $\dfrac{dy}{dx}$ if $y = 3 \ln x$

8. $\dfrac{dy}{dx}$ if $y = x^2 \ln x$

4) $D_x \ln(3x^3+2x) = \dfrac{9x^2+2}{3x^2+2x}$

5) $D_x \ln(x-4)^3 = \dfrac{1}{(x-4)^{\cancel{3}}} \cdot 3\cancel{(x-4)^2}$

$= \dfrac{3}{(x-4)}$

6) $D_x \ln\sqrt{3x-2} = D_x \ln(3x-2)^{1/2}$

$D_x \frac{1}{2} \ln(3x-2) = \frac{1}{2} \dfrac{3}{3x-2}$

$= \dfrac{3}{2(3x-2)}$

مشتقة الضرب
الاول x مشتقة الثاني + الثاني x مشتقة الاول

8) $y = x^2 \ln x$

$\dfrac{dy}{dx} = x^2 \dfrac{1}{x} + 2x \ln x$

$\dfrac{dy}{dx} = x + 2x \ln x = x(1 + 2\ln x)$

In Problems 15–26, find the integrals

18. $\int \frac{z}{2z^2+8}\,dz$

20. $\int \frac{-1}{x(\ln x)^2}\,dx$

22. $\int_0^1 \frac{t+1}{2t^2+4t+3}\,dt$

24. $\int \frac{x^2+x}{2x-1}\,dx$

$$\int \frac{f'}{f}\,dx = \ln|f|$$

18) $\int \frac{z}{2z^2+8}\,dz$

$u = 2z^2+8$

$du = 4z\,dz$

$dz = \frac{du}{4z}$

$$\int \frac{\cancel{z}}{u}\,\frac{du}{4\cancel{z}} = \frac{1}{4}\int \frac{du}{u}$$

$$= \frac{1}{4}\ln u = \frac{1}{4}\ln|2z^2+8| + C$$

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20) $\int \frac{-1}{x(\ln x)^2}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

$$\int \frac{-1}{\cancel{x}\,u^2}\,\cancel{x}\,du = -\int \frac{du}{u^2} = -\int u^{-2}\,du$$

$$= -\frac{u^{-1}}{-1} = u^{-1} = \frac{1}{u} = \frac{1}{\ln x} + C$$

24) $\int \frac{x^2+x}{2x-1}\,dx$

قسمة طويلة

$$\begin{array}{r|l} & \frac{x}{2} + \frac{3}{4} \\ \hline 2x-1 & x^2+x \\ & -x^2 + \frac{x}{2} \\ \hline & \frac{3x}{2} \\ & -\frac{3}{2}x + \frac{3}{4} \\ \hline & \frac{3}{4} \end{array}$$

$$\int \frac{x}{2} + \frac{3}{4} + \frac{3}{4(2x-1)}\,dx$$

$$= \frac{x^2}{2\cdot 2} + \frac{3}{4}x + \frac{3}{4\cdot 2}\int \frac{2}{2x-1}\,dx$$

$$= \frac{x^2}{4} + \frac{3}{4}x + \frac{3}{8}\ln|2x-1| + c$$

In Problems 27–30, use Theorem A to write the expressions as the logarithm of a single quantity.

27. $2\ln(x+1) - \ln x$ **28.** $\frac{1}{2}\ln(x-9) + \frac{1}{2}\ln x$

27) $\ln(x+1)^2 - \ln x = \ln \frac{(x+1)^2}{x}$

28) $\ln(x-9)^{1/2} + \ln x^{1/2} = \ln (x-9)^{1/2}\cdot x^{1/2}$

$\ln\sqrt{(x-9)\,x}$

In Problems 31–34, find dy/dx by logarithmic differentiation (see Example 8).

31. $y = \dfrac{x+11}{\sqrt{x^3-4}}$

$$\ln y = \ln \frac{x+11}{\sqrt{x^3-4}}$$

$$= \ln(x+11) - \ln(x^3-4)^{1/2}$$

$$\ln y = \ln(x+11) - \frac{1}{2}\ln(x^3-4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{1}{2}\,\frac{3x^2}{x^3-4}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{3x^2}{2(x^3-4)}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4}\right]$$

$$\frac{dy}{dx} = \frac{x+11}{\sqrt{x^3+4}}\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4)}\right]$$