CHAPTER 6

# Transcendental Functions

لوغاريتمات
الدوال الاسية
الدوال المثلثية
الدوال العكسية

## 6.1 The Natural Logarithm Function (ln)

$\log_2 8 = 3$ $\quad 2^3 = 8$

ln $\cancel{\log}_e m = a$ $\quad e^a = m$ $\quad e^x = 4$ $\quad \ln 4 = x$

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Differential $\qquad$ مشتقة ln

$$\boxed{D_x \ln x = \frac{1}{x}}$$

$$D_x \ln \text{دالة} = \frac{\text{مشتقة الدالة}}{\text{الدالة نفسها}} \qquad D_x \ln u = \frac{1}{u} D_x u$$

$$D_x \ln(3x+1) = \frac{3}{3x+1}$$

$$\frac{d}{dx} \ln \sin x = \frac{\cos x}{\sin x}$$

مثال

find the derevative

a) $\ln 81$

$\frac{d}{dx} \ln 81 = 0$

b) $\ln(x^3+4x+10)$

$\frac{d}{dx} \ln(x^3+4x+10) = \frac{3x^2+4}{x^3+4x+10}$

**EXAMPLE 1** Find $D_x \ln \sqrt{x}$. → المشتقة / الدالة

$U = \sqrt{x} = x^{1/2}$ $\qquad D_x U = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$D_x \ln U = \frac{D_x U}{U} = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{x}} = \frac{1}{2x}$$

**EXAMPLE 2** Find $D_x \ln(x^2 - x - 2)$.

$D_x \ln(x^2 - x - 2) =$ المشتقة / الدالة

$$= \frac{2x-1}{x^2-x-2}$$

**EXAMPLE 3** Show that

$$\boxed{D_x \ln|x| = \frac{1}{x}, \qquad x \neq 0}$$

$$D_x \ln|x| = \begin{cases} D_x \ln x = \frac{1}{x} \\ D_x \ln -x = \frac{-1}{-x} = \frac{1}{x} \end{cases}$$

دائماً الجواب $\frac{1}{x}$

تكامل اللوغاريتم الطبيعي

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \frac{3}{4x+1}\,dx = 3\int \frac{1}{4x+1}\,dx = \frac{3}{4}\ln|4x+1| + c$$

$$\int \frac{1}{u}\,du = \int \frac{\text{مشتقة}}{\text{الدالة الأصلية}}\,dx = \ln(\underset{u}{\text{الدالة}})$$

$$\int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c$$

**EXAMPLE 4** Find $\int \frac{5}{2x+7}\,dx$.

$$5\int \frac{1}{2x+7}\,dx = \frac{5}{2}\ln|2x+7| + c$$

ملاحظة

$u = 2x+7$ $\qquad \frac{du}{2} = dx$

نعوض كل $x$ بـ $u$

$$\int \frac{5}{2x+7}\,dx = \int \frac{5}{u}\,\frac{du}{2} = \frac{5}{2}\int \frac{du}{u}$$

$$\frac{5}{2}\ln u + c = \frac{5}{2}\ln|2x+7| + c$$

**EXAMPLE 5** Evaluate $\int_{-1}^{3} \frac{x}{10-x^2}\,dx$.

نضرب البسط والمقام بـ -2

$$\frac{1}{-2}\int \frac{-2x}{10-x^2}\,dx = -\frac{1}{2}\ln|10-x^2|$$

$u = 10 - x^2 \qquad du = -2x\,dx \qquad dx = \frac{du}{-2x}$

نعوض بدلاله u

$$\int \frac{x}{10-x^2}\,dx = \int \frac{\cancel{x}}{u}\,\frac{du}{-2\cancel{x}} = \frac{1}{-2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|10-x^2|\Big]_{-1}^{3}$$

$$= -\frac{1}{2}\left[\ln|10-3^2| - \ln|10-(-1)^2|\right]$$

$$= -\frac{1}{2}\left[\ln 1 - \ln 9\right] = -\frac{1}{2}\ln 1 + \frac{1}{2}\ln 9$$

**EXAMPLE 6** Find $\int \frac{x^2-x}{x+1}\,dx$.

لان درجه البسط اكبر من المقام
نقسم قسمه طويله

$$\frac{\text{الباقي}}{\text{المقسوم عليه}} + \text{الناتج}$$

$$\begin{array}{r} x-2 \\ x+1 \overline{\big) x^2 - x} \\ \underline{-x^2 \mp x} \\ -2x \\ \underline{\mp 2x \mp 2} \\ 2 \end{array}$$

$$\int \frac{x^2-x}{x+1}\,dx = \int\left[(x-2) + \frac{2}{x+1}\right]dx$$

$$= \int (x-2)\,d + 2\int \frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2} - 2x + 2\ln|x+1| + C$$

**EXAMPLE 7** Find $dy/dx$ if $y = \ln \sqrt[3]{(x-1)/x^2}$, $x > 1$.

احسب المشتقة

$$y = \ln \sqrt[3]{\frac{x-1}{x^2}} \qquad x > 1$$

نستخدم قواعد اللوغاريتمات

$$\ln a^n = n \ln a \qquad \ln ab = \ln a + \ln b$$

$$\ln \frac{a}{b} = \ln a - \ln b$$

$$y = \ln\left(\frac{x-1}{x^2}\right)^{1/3} = \frac{1}{3}\ln\frac{x-1}{x^2}$$

$$y = \frac{1}{3}\left(\ln(x-1) - \ln x^2\right)$$

$$y = \frac{1}{3}\left(\ln(x-1) - 2\ln x\right)$$

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x-1} - 2\,\frac{1}{x}\right]$$

$$= \frac{1}{3}\left[\frac{x}{x(x-1)} - \frac{2(x-1)}{x(x-1)}\right]$$

$$= \frac{1}{3}\left[\frac{x-2x+2}{x(x-1)}\right] = \frac{-x+2}{3x(x-1)}$$

**EXAMPLE 8** Differentiate $y = \dfrac{\sqrt{1-x^2}}{(x+1)^{2/3}}$. $= \dfrac{(1-x^2)^{1/2}}{(x+1)^{2/3}}$

$$\frac{dy}{dx} = \frac{f}{g} = \frac{gf' - fg'}{g^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}(x+1)^{2/3}(1-x^2)^{-1/2}(2x) - \frac{2}{3}(1-x^2)^{1/2}(x+1)^{-1/3}}{(x+1)^{4/3}}$$

بسط اكل

حل آخر ندخل ln لطرفي المعادلة

$3y^2 = 4x+5$

$6y\frac{dy}{dx} = 4$

$$\ln y = \ln \frac{\sqrt{1-x^2}}{(x+1)^{2/3}}$$

$$\ln y = \ln(1-x^2)^{1/2} - \ln(x+1)^{2/3}$$

$$\ln y = \frac{1}{2}\ln(1-x^2) - \frac{2}{3}\ln(x+1)$$

نشتق الطرفين

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\,\frac{-2x}{1-x^2} - \frac{2}{3}\,\frac{1}{x+1}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-x}{1-x^2} - \frac{2}{3(x+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = {}^{\times 3}\frac{-x}{(1+x)(1-x)} - {}^{(1-x)}\frac{2}{3(x+1)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-3x-2+2x}{3(1+x)(1-x)} = \frac{-2-x}{3(1-x^2)}$$

$$\frac{dy}{dx} = -y\,\frac{(2+x)}{3(1-x^2)}$$

$$\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}\,\sqrt{1-x^2}} =$$

$$\frac{dy}{dx} = \frac{-\sqrt{1-x^2}}{3\;(x+1)^{2/3}}\;\frac{(2+x)}{(1-x^2)}$$

$$\frac{dy}{dx} = \frac{-(x+2)}{3(x+1)^{2/3}(1-x^2)^{1/2}}$$

**EXAMPLE 9** Evaluate $\int \tan x\, dx$.

$$\frac{-\sin x}{\cos x}$$

$$-\int -\frac{\sin x}{\cos x}\, dx = -\ln|\cos x| + C$$

$$\int \frac{\sin x}{\cos x}\, dx = \int \frac{\cancel{\sin x}}{u}\,\frac{du}{\cancel{-\sin x}} = -\int \frac{1}{u}\, du = -\ln|\cos x|$$

$\cos x = u \qquad du = -\sin x\, dx \qquad dx = \dfrac{du}{-\sin x}$

**EXAMPLE 10** Evaluate $\int \sec x \csc x\, dx$.

$\sec x \csc x = \tan x + \cot x$

$$\int \tan x\, dx + \int \cot x\, dx \qquad \int \frac{1}{u}\, du$$

$$\int \frac{\sin x}{\cos x}\, dx + \int \frac{\cos x}{\sin x}$$

$$-\ln|\cos x| + \ln|\sin x| + C$$

# Problem Set 6.1

*In Problems 3–14, find the indicated derivative (see Examples 1 and 2). Assume in each case that x is restricted so that ln is defined.*

**3.** $D_x \ln(x^2 + 3x + \pi)$

**5.** $D_x \ln(x - 4)^3$

3) $\dfrac{2x+3}{x^2+3x+\pi}$

5) $\dfrac{3(x-4)^2}{(x-4)^3} = \dfrac{3}{(x-4)}$

**7.** $\dfrac{dy}{dx}$ if $y = 3 \ln x$

7) $y' = 3\,\dfrac{1}{x} = \dfrac{3}{x}$

**9.** $\dfrac{dz}{dx}$ if $z = x^2 \ln x^2 + (\ln x)^3$ $= 2x^2 \ln x + (\ln x)^3$

9) $2x^2\,\dfrac{1}{x} + 4x \ln x + 3(\ln x)^2\,\dfrac{1}{x}$

$2x + 4x \ln x + \dfrac{3}{x}(\ln x)^2$

**13.** $f'(81)$ if $f(x) = \ln \sqrt[3]{x}$ $= \ln x^{1/3} = \frac{1}{3} \ln x$

$f'(x) = \dfrac{1}{3}\,\dfrac{1}{x} = \dfrac{1}{3x}$

$f'(81) = \dfrac{1}{3 \cdot 81} = \dfrac{1}{243}$

**11.** $g'(x)$ if $g(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ $\quad \to (x^2+1)^{1/2}$

$$g'(x) = \frac{1 + \frac{1}{\cancel{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x}{x + \sqrt{x^2+1}}$$

$$= \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\, 1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

$$= \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$$

$$= \frac{\frac{\cancel{x + \sqrt{x^2+1}}}{\sqrt{x^2+1}}}{\cancel{x + \sqrt{x^2+1}}}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

*In Problems 31–34, find $dy/dx$ by logarithmic differentiation (see Example 8).*

**31.** $y = \dfrac{x + 11}{\sqrt{x^3 - 4}}$

$$\ln y = \ln \frac{x+11}{\sqrt{x^3-4}}$$

$$= \ln(x+11) - \ln(x^3-4)^{1/2}$$

$$\ln y = \ln(x+11) - \frac{1}{2}\ln(x^3-4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{1}{2}\frac{3x^2}{x^3-4}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{3x^2}{2(x^3-4)}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4)}\right]$$

$$\frac{dy}{dx} = \frac{x+11}{\sqrt{x^3+4}}\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4)}\right]$$

**33.** $y = \dfrac{\sqrt{x+13}}{(x-4)\sqrt[3]{2x+1}}$

$$\ln y = \ln (x+3)^{1/2} - \ln (x-4) - \ln (2x+1)^{1/3}$$

$$\ln y = \frac{1}{2}\ln(x+13) - \ln(x-4) - \frac{1}{3}\ln(2x+1)$$

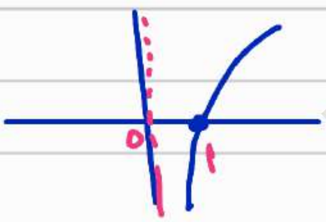
$$\frac{1}{y}y' = \frac{1}{2}\,\frac{1}{x+13} - \frac{1}{x-1} - \frac{1}{3}\,\frac{2}{2x+1}$$

$$y' = \frac{1}{2(x+13)} - \frac{1}{(x-1)} - \frac{2}{3(2x+1)} \cdot y$$

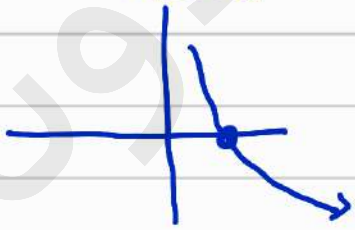
$$y' = \left[\frac{1}{2(x+13)} - \frac{1}{x-1} - \frac{2}{3(2x+1)}\right]\frac{\sqrt{x+13}}{(x-4)\sqrt[3]{2x+1}}$$



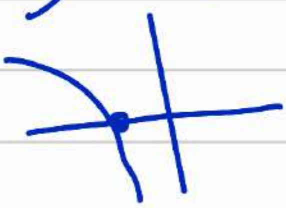
$\ln x$



$-\ln x$

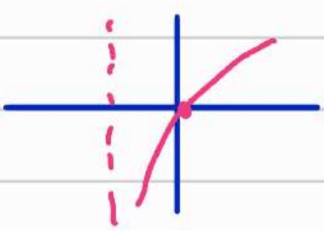


$\ln(-x)$

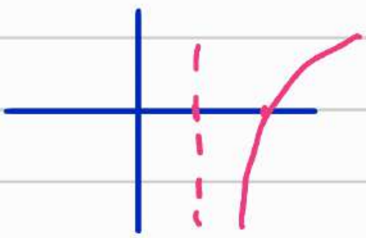


$4\ln x$

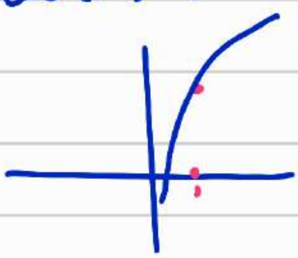
$\ln(x+1)$



$\ln(x-1)$



$\ln(x)+2$

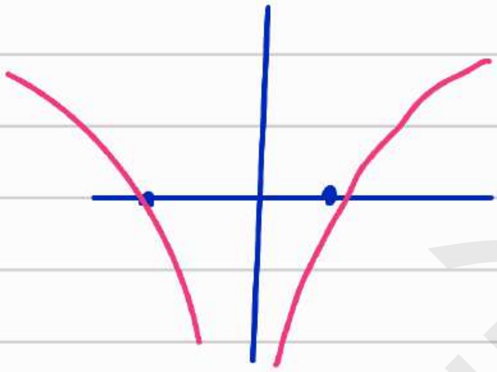


*In Problems 35–38, make use of the known graph of $y = \ln x$ to sketch the graphs of the equations.*

**35.** $y = \ln|x|$ **36.** $y = \ln\sqrt{x}$

**37.** $y = \ln\left(\frac{1}{x}\right)$ **38.** $y = \ln(x - 2)$

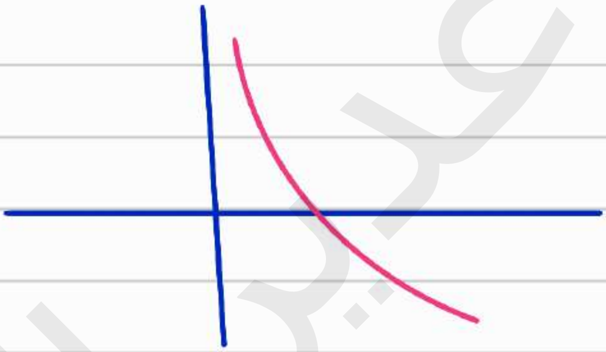
35) $\ln|x| \begin{cases} \ln x \\ \ln -x \end{cases}$



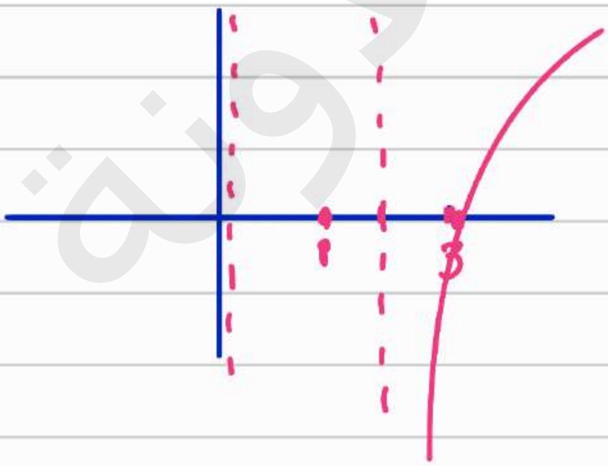
73) $y = \ln\left(\frac{1}{x}\right)$

$y = \ln x^{-1}$

$= -\ln x$



38)



# Problem Set 6.1

*In Problems 3–14, find the indicated derivative (see Examples 1 and 2). Assume in each case that x is restricted so that ln is defined.*

3. $D_x \ln(x^2 + 3x + \pi)$
4. $D_x \ln(3x^3 + 2x)$
5. $D_x \ln(x - 4)^3$
6. $D_x \ln \sqrt{3x - 2}$
7. $\frac{dy}{dx}$ if $y = 3 \ln x$
8. $\frac{dy}{dx}$ if $y = x^2 \ln x$

4) $D_x \ln(3x^3 + 2x) = \dfrac{9x^2 + 2}{3x^2 + 2x}$

5) $D_x \ln(x-4)^3 = \dfrac{1}{(x-4)^{\cancel{3}}} \cdot 3\cancel{(x-4)^2}$

$= \dfrac{3}{(x-4)}$

6) $D_x \ln\sqrt{3x-2} = D_x \ln(3x-2)^{1/2}$

$D_x \frac{1}{2} \ln(3x-2) = \frac{1}{2} \dfrac{3}{3x-2}$

$= \dfrac{3}{2(3x-2)}$

مشتقة الضرب
اول × مشتقة الثاني + الثاني × مشتقة الاول

8) $y = x^2 \ln x$

$\dfrac{dy}{dx} = x^2 \dfrac{1}{x} + 2x \ln x$

$\dfrac{dy}{dx} = x + 2x \ln x = x(1 + 2\ln x)$

*In Problems 15–26, find the integrals*

18. $\int \frac{z}{2z^2+8}\,dz$

20. $\int \frac{-1}{x(\ln x)^2}\,dx$

22. $\int_0^1 \frac{t+1}{2t^2+4t+3}\,dt$

24. $\int \frac{x^2+x}{2x-1}\,dx$

$$\int \frac{f'}{f}\,dx = \ln|f|$$

18)

$$\int \frac{z}{2z^2+8}\,dz$$

$u = 2z^2+8$

$du = 4z\,dz$

$dz = \frac{du}{4z}$

$$\int \frac{\cancel{z}}{u}\,\frac{du}{4\cancel{z}} = \frac{1}{4}\int \frac{du}{u}$$

$$= \frac{1}{4}\ln u = \frac{1}{4}\ln|2z^2+8| + C$$

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20)

$$\int \frac{-1}{x(\ln x)^2}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

$$\int \frac{-1}{\cancel{x}\,u^2}\,\cancel{x}\,du = -\int \frac{du}{u^2} = -\int u^{-2}\,du$$

$$= -\frac{u^{-1}}{-1} = u^{-1} = \frac{1}{u} = \frac{1}{\ln x} + C$$

قسمة طويلة

24) $\int \frac{x^2+x}{2x-1}\,dx$

$$\begin{array}{r|l} & \frac{x}{2}+\frac{3}{4} \\ 2x-1 & x^2+x \\ & -x^2+\frac{x}{2} \\ \hline & \frac{3x}{2} \\ & -\frac{3}{2}x+\frac{3}{4} \\ \hline & \frac{3}{4} \end{array}$$

$$\int \frac{x}{2}+\frac{3}{4}+\frac{3}{4(2x-1)}\,dx$$

$$= \frac{x^2}{2\cdot 2}+\frac{3}{4}x+\frac{3}{4\cdot 2}\int \frac{2}{2x-1}\,dx$$

$$= \frac{x^2}{4}+\frac{3}{4}x+\frac{3}{8}\ln|2x-1|+c$$

*In Problems 27–30, use Theorem A to write the expressions as the logarithm of a single quantity.*

**27.** $2\ln(x+1)-\ln x$ **28.** $\frac{1}{2}\ln(x-9)+\frac{1}{2}\ln x$

27) $\ln(x+1)^2-\ln x = \ln\frac{(x+1)^2}{x}$

28) $\ln(x-9)^{1/2}+\ln x^{1/2} = \ln(x-9)^{1/2}\cdot x^{1/2}$

$\ln\sqrt{(x-9)x}$

*In Problems 31–34, find $dy/dx$ by logarithmic differentiation (see Example 8).*

**31.** $y = \dfrac{x+11}{\sqrt{x^3-4}}$

$$\ln y = \ln \frac{x+11}{\sqrt{x^3-4}}$$

$$= \ln(x+11) - \ln(x^3-4)^{1/2}$$

$$\ln y = \ln(x+11) - \frac{1}{2}\ln(x^3-4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{1}{2}\,\frac{3x^2}{x^3-4}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+11} - \frac{3x^2}{2(x^3-4)}$$

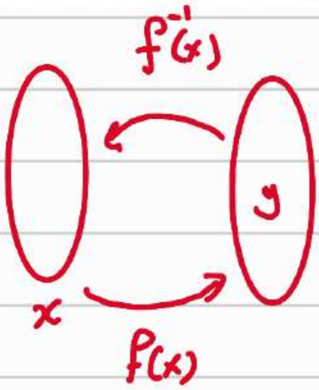
$$\frac{dy}{dx} = y\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4)}\right]$$

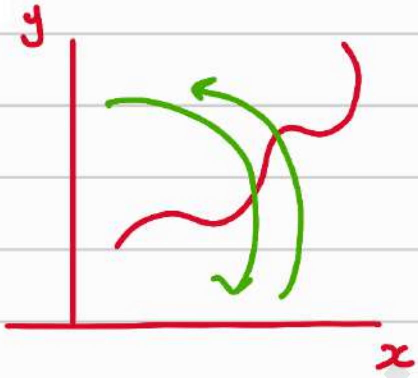
$$\frac{dy}{dx} = \frac{x+11}{\sqrt{x^3+4}}\left[\frac{1}{x+11} - \frac{3x^2}{3(x^3-4)}\right]$$

CHAPTER 6 Transcendental Functions

## 6.2 Inverse Functions and Their Derivatives

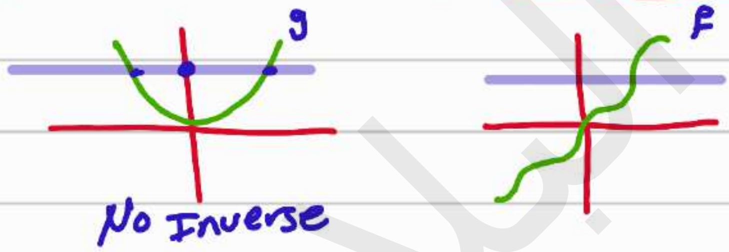
الدوال العكسيه ومشتقاتها




$y = 3x + 1 \quad \longrightarrow \quad x = \frac{y-1}{3}$

$f^{-1}(y)$




اذا قطع الخط الافقي الدالة اكثر من مرة اذاً لا تكون لها معكوس

متزايده دائماً او متناقصه دائماً

لها معكوس

للتأكد ان الداله دائماً متناقصه او متزايده نشتق الدالة وندرس اشارتها

مثال :- هل هذه الدالة لها معكوس $f(x) = -x^3 + 5$

$f'(x) = -3x^2$


الاشتقاق دائماً سالبه اذا الداله متناقصة (نعم لها معكوس)

**EXAMPLE 1** Show that $f(x) = x^5 + 2x + 1$ has an inverse.

$f'(x) = 5x^4 + 2$ دائماً موجبه

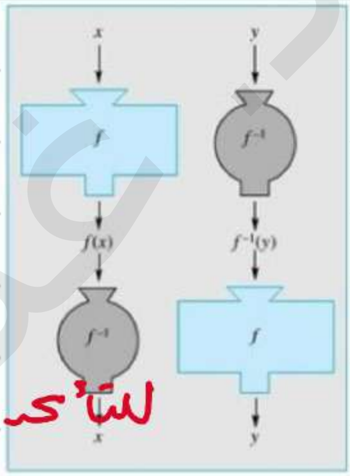
$5x^2 + 2 > 0$

**Theorem A**
If $f$ is strictly monotonic on its domain, then $f$ has an inverse.

الدالة دائماً متزايدة يعني لديها معكوس

Yes it has inverse

$$\boxed{f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y}$$




**EXAMPLE 2** Show that $\underset{}{\overset{y}{f(x)}} = 2x + 6$ has an inverse, find a formula for $\underset{x}{f^{-1}(y)}$, and verify the results in the box above.

للتأكد ان الدالة لها معكوس نشتق وندرس الاشارة

$f'(x) = 2$ دائماً موجبه لها معكوس

it has inverse

$y = 2x + 6$

$2x = y - 6$

$x = \dfrac{y-6}{2}$

$f^{-1}(y) = x = \dfrac{y-6}{2}$

$f^{-1}(\cancel{f(x)}^{\,y}) = \dfrac{y-6}{2} = \dfrac{(2x+\cancel{6}) - \cancel{6}}{2} = \dfrac{2x}{2} = x$

$f(f^{-1}(y)) = 2(f^{-1}(y)) + 6 = \cancel{2}\left(\dfrac{y-\cancel{6}}{\cancel{2}}\right) + \cancel{6} = y$

**EXAMPLE 3** Find a formula for $f^{-1}(x)$ if $y = f(x) = x/(1-x)$.

① نبدل $f(x)$ ب $y$

$$\cancel{f}(x) = \frac{x}{1-x}$$

② نكتب المعادلة بدلالة $x$

$$\frac{y}{1} = \frac{x}{1-x}$$

$$y(1-x) = x \Rightarrow y - yx = x$$

$$y = x + yx \Rightarrow \frac{y}{1+y} = \frac{x(1+y)}{1+y}$$

$$\boxed{x = \frac{y}{1+y}}$$

③ نبدل قيمة $x$ ب $f^{-1}(y)$

$$f^{-1}(y) = \frac{y}{1+y}$$

④ نبدل كل قيم $y$ ب $x$

$$f^{-1}(x) = \frac{x}{1+x}$$

**Theorem B** **Inverse Function Theorem**

Let $f$ be differentiable and strictly monotonic on an interval $I$. If $f'(x) \neq 0$ at a certain $x$ in $I$, then $f^{-1}$ is differentiable at the corresponding point $y = f(x)$ in the range of $f$ and

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

مشتقة الدالة العكسية بالنسبة لـ $y$ = $\frac{1}{\text{المشتقة بالنسبة لـ } x}$

The conclusion to Theorem B is often written symbolically as

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

**EXAMPLE 4** Let $y = f(x) = x^5 + 2x + 1$, as in Example 1. Find $(f^{-1})'(4)$.

المطلوب حساب مشتقة الدالة العكسية لـ y

$$(f^{-1})'(y) = \frac{1}{f'(x)} \qquad y=4 \quad x=1$$

$$(f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{5x^4+2}$$

$$= \frac{1}{5(1)^4+2} = \frac{1}{7}$$

# Problem Set 6.2

*In Problems 7–14, show that $f$ has an inverse by showing that it is strictly monotonic (see Example 1).*

**7.** $f(x) = -x^5 - x^3$

$f'(x) = -5x^4 - 3x^2 \Longrightarrow$

دائماً موجبة

دائماً

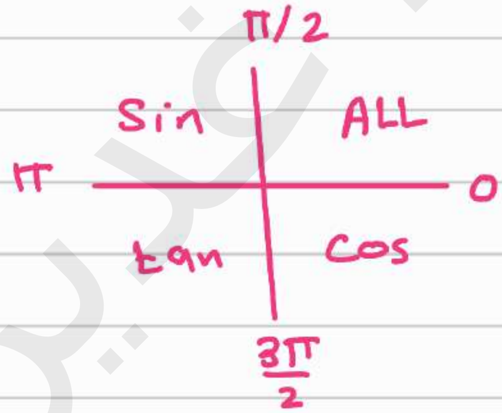
المشتقة دائماً سالبة

strictly decreasing

it has inverse

$f'(x)$ - - - - - - - - - - -

**9.** $f(\theta) = \cos\theta, 0 \le \theta \le \pi$




$f'(\theta) = -\sin\theta$

$f'(x)$ – – – – – – – –

0 π

strictly decreasing

it has inverse

**11.** $f(z) = (z-1)^2, z \ge 1$

$f'(x) = 2(z-1)$

المشتقة دائماً موجبة

+ + + +

1 ∞

strictly increasing

has inverse

*In Problems 15–28, find a formula for $f^{-1}(x)$ and then verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (see Examples 2 and 3).*

**17.** $f(x) = \sqrt{x+1}$

$x+1 \geq 0 \qquad x \geq 1 \qquad y \geq 0$

$y = \sqrt{x+1} \qquad y^2 = x+1 \qquad x = y^2 - 1$

$\boxed{f^{-1}(x) = x^2 - 1} \qquad x \geq 0$

$f^{-1}(f(x)) = \sqrt{x+1}^2 - 1 = x + 1 - 1 = x$

$f(f^{-1}(x)) = \sqrt{x^2 - 1 + 1} = \sqrt{x^2} = x$

**19.** $f(x) = -\dfrac{1}{x-3}$

$y = -\dfrac{1}{x-3} \qquad x - 3 = -\dfrac{1}{y} \qquad x = 3 - \dfrac{1}{y}$

$\boxed{f^{-1}(x) = 3 - \dfrac{1}{x}}$

$f^{-1}(f(x)) = 3 - \dfrac{1}{\frac{-1}{x-3}} = 3 + (x-3) = x$

$f(f^{-1}(x)) = -\dfrac{1}{\cancel{3} - \frac{1}{x} - \cancel{3}} = -\dfrac{1}{-\frac{1}{x}} = x$

**21.** $f(x) = 4x^2, x \leq 0$

$y = 4x^2 \qquad \sqrt{\dfrac{y}{4}} = x \qquad x = \sqrt{\dfrac{y}{4}}$

$x = -\dfrac{\sqrt{y}}{2}$

$\boxed{f^{-1}(x) = -\dfrac{\sqrt{x}}{2}}$

$$f^{-1}(f(x)) = -\frac{\sqrt{4x^2}}{2} = -\frac{\cancel{2}\sqrt{x^2}}{\cancel{2}} = x$$

$$f(f^{-1}(x)) = 4\left(-\frac{\sqrt{x}}{2}\right)^2 = \cancel{4}\,\frac{x}{\cancel{4}} = x$$

**23.** $f(x) = (x-1)^3$

$$y = (x-1)^3 \qquad \sqrt[3]{y} = x-1$$

$$x = 1 + \sqrt[3]{y} \qquad \boxed{f^{-1}(x) = 1 + \sqrt[3]{x}}$$

$$f^{-1}(f(x)) = 1 + \sqrt[3]{(x-1)^{\cancel{3}}} = 1 + x - 1 = x$$

$$f(f^{-1}(x)) = (\cancel{1} + \sqrt[3]{x} - \cancel{1})^3 = (\sqrt[3]{x})^3 = x$$

# Problem Set 6.2

*In Problems 7–14, show that $f$ has an inverse by showing that it is strictly monotonic (see Example 1).*

**7.** $f(x) = -x^5 - x^3$ **8.** $f(x) = x^7 + x^5$

7) $f(x) = -x^5 - x^3$

$f'(x) = -5x^4 - 3x^2 < 0$ strictly decrease

موجب موجب

it has inverse

8) $f(x) = x^7 + x^5$

$f'(x) = 7x^6 + 5x^4 > 0$ strictly increasing

$+$ $+$

it has inverse

*In Problems 15–28, find a formula for $f^{-1}(x)$ and then verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (see Examples 2 and 3).*

**15.** $f(x) = x + 1$ **16.** $f(x) = -\frac{x}{3} + 1$

**17.** $f(x) = \sqrt{x + 1}$ **18.** $f(x) = -\sqrt{1 - x}$

**19.** $f(x) = -\frac{1}{x - 3}$ **20.** $f(x) = \sqrt{\frac{1}{x - 2}}$

16) $f(x) = -\frac{x}{3} + 1$

$y = -\frac{x}{3} + 1$ $\qquad$ $-\frac{x}{3} = y - 1$

$x = -3(y-1)$ $\qquad$ $x = -3y + 3$

$f^{-1}(y) = -3y + 3$

$$\boxed{f^{-1}(x) = -3x + 3}$$

$$f^{-1}(f(x)) = -3\left(-\frac{x}{3} + 1\right) + 3 = x - 3 + 3 = x$$

$$f(f^{-1}(x)) = -\frac{-3x+3}{3} + 1 = +x - 1 + 1 = x$$

---

18) $f(x) = -\sqrt{1-x}$ $\qquad$ $(y)^2 = \left(-\sqrt{1-x}\right)^2$

$y^2 = 1 - x$ $\qquad$ $x = 1 - y^2$

$f^{-1}(y) = 1 - y^2$ $\qquad$ $f^{-1}(x) = 1 - x^2$

$$f^{-1}(f(x)) = 1 - \left(\sqrt{1-x}\right)^2 = 1 - (1 - x) = 1 - 1 + x = x$$

$$f(f^{-1}(x)) = \sqrt{1 - (1 - x^2)} = \sqrt{1 - 1 + x^2} = \sqrt{x^2} = x$$

19) $y = \frac{-1}{x-3}$ $(x-3)y = -1$

$x - 3 = \frac{-1}{y}$ $x = 3 - \frac{1}{y}$

$f^{-1}(y) = 3 - \frac{1}{y}$

$f^{-1}(x) = 3 - \frac{1}{x}$

*In Problems 37–40, find $(f^{-1})'(2)$ by using Theorem B (see Example 4). Note that you can find the x corresponding to $y = 2$ by inspection.*

**37.** $f(x) = 3x^5 + x - 2$ **38.** $f(x) = x^5 + 5x - 4$

$(f^{-1})'(y) = \frac{1}{f'(x)}$

---

37) $y = 2$ $x = 1$

$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{16}$

$f'(x) = 15x^4 + 1$ $f'(1) = 15 + 1 = 16$

---

38) $y = 2$ $x = 1$

$f'(x) = 5x^4 + 5$ $f'(1) = 10$

$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{10}$

## 6.3
# The Natural Exponential Function

$e = 2.718 \ldots$

ln

$e^x = \exp(x)$

$e^x = y \quad \log_e y = x$

$\ln x = \log_e x$

$e^{\ln x} = x \qquad \ln e^x = x$

$e^3 e^5 = e^8 \qquad e^8 \div e^3 = e^5$

$e^x$ مشتقة

$D_x e^x = e^x$

$D_x e^{3x} = 3e^{3x}$

$D_x \ln x = \frac{1}{x}$

$D_x \ln(f(x)) = \frac{f'}{f}$

$\boxed{D_x e^x = e^x} \qquad \boxed{D_x e^f = f' e^f}$

**EXAMPLE 1** Find $D_x e^{\sqrt{x}}$.

$D_x \sqrt{x} =$

$D_x x^{1/2}$

$\frac{1}{2} x^{-1/2}$

$\frac{1}{2 x^{1/2}} = \frac{1}{2\sqrt{x}}$

$$D_x e^{\sqrt{x}} = \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

**EXAMPLE 2** Find $D_x e^{x^2 \ln x}$.

$$D_x e^{x^2 \ln x} = x(1+2\ln x) e^{x^2 \ln x}$$

$$D_x x^2 \ln x = x^2 \frac{1}{x} + 2x \ln x = x + 2x \ln x$$

$$x(1+2\ln x)$$

أسئلة اضافية

$*$ $D_x e^{-2\ln x} = -2\frac{1}{x} e^{-2\ln x} = -\frac{2}{x} e^{-2\ln x}$

$*$ $D_x \cancel{\ln e}^{-2x-3} = D_x(-2x-3) = -2$

$*$ $D_x e^{\ln x^2} = D_x x^2 = 2x$

$-4e^{-2}$

**EXAMPLE 3** Let $f(x) = xe^{x/2}$. Find where $f$ is increasing and decreasing, and where it is concave upward and downward. Also, identify all extreme values and points of inflection. Then, sketch the graph of $f$.

① حساب المشتقة الاولى و ايجاد اصفارها و اشاراتها

$$f'(x) = \frac{x}{2}e^{x/2} + 1\,e^{x/2} = \left(\frac{x+2}{2}\right)e^{x/2}$$

$$f'(x) = \left(\frac{x+2}{2}\right)e^{x/2}$$

ندرس اشاراتها

دائماً موجبة

$$\left(\frac{x+2}{2}\right) = 0 \qquad x+2=0 \qquad x=-2$$

$$- \; - \; - \; - \; - \; \underset{-2}{|} \; + \; + \; +$$

فترة التناقص $(-\infty, -2)$ فترة التزايد $(-2, \infty)$

والقيمة الصغرى $\left(-2, -\frac{2}{e}\right)$ ← $-0.7$

② نحسب المشتقة الثانية

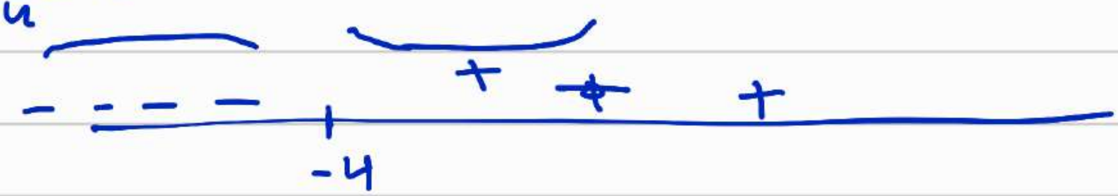
$$f' = \left(\frac{x+2}{2}\right)e^{x/2}$$

$$f'' = \left(\frac{x+2}{2}\right)\frac{1}{2}e^{x/2} + \frac{1}{2}e^{x/2}$$

$$f'' = \left(\frac{(x+2)}{4} + \frac{1 \cdot 2}{2 \cdot 2}\right)e^{x/2} = \frac{x+4}{4}e^{x/2}$$

دائماً موجب

$\frac{x+4}{4}=0 \qquad x+4=0 \qquad x=-4$

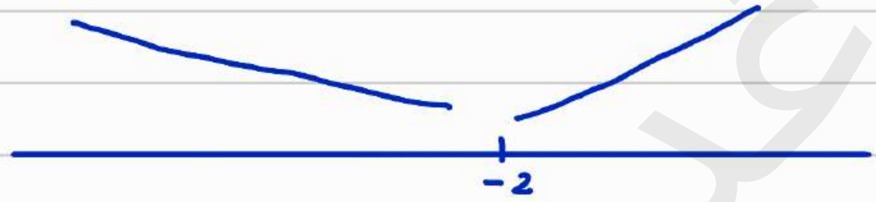


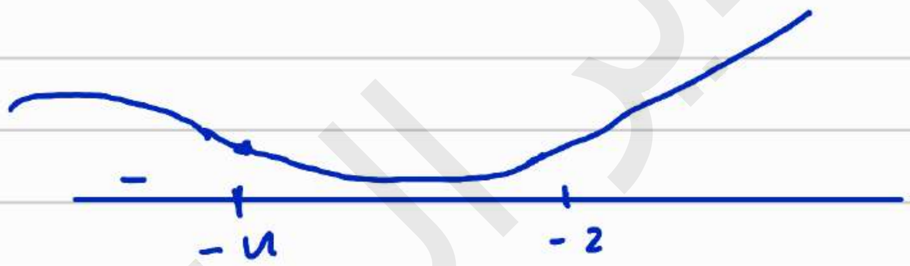

upward concave $(-4,\infty)$ فترات التقعر للأعلى

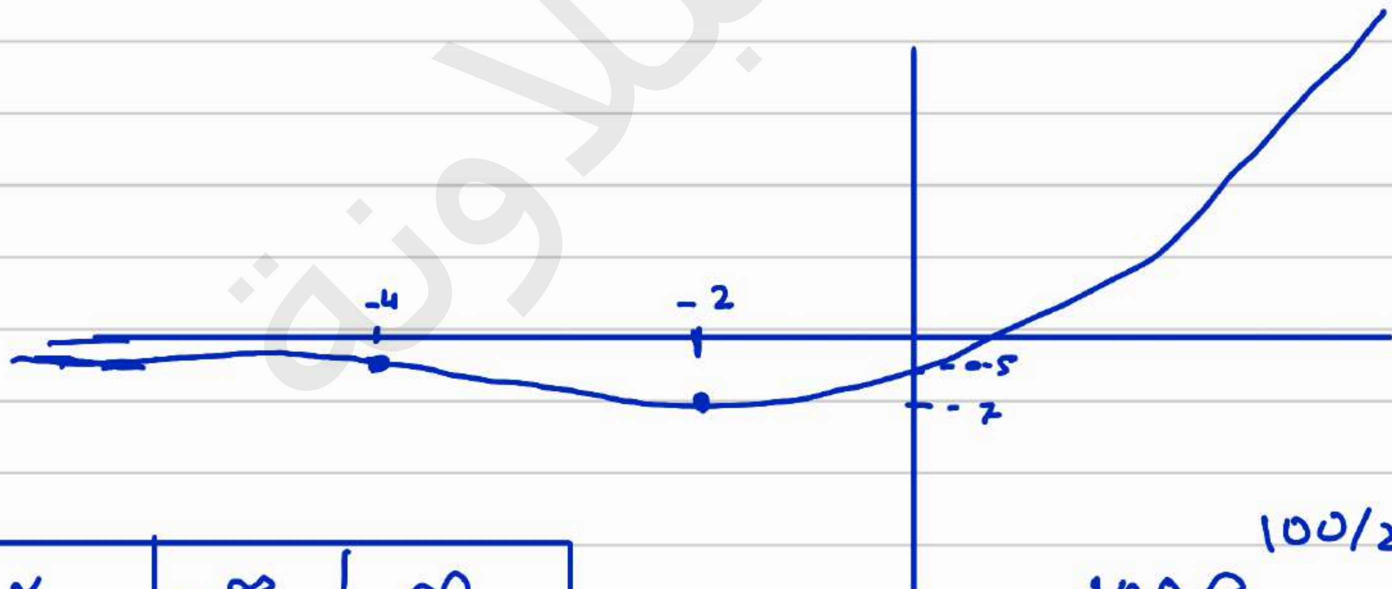
concave downward $(-\infty,-4)$ التقعر للأسفل

inflection $(4,-4e^{-2})$ نقطة انعطاف

$-0.54$




$100e^{100/2}$

| $x$ | $-\infty$ | $\infty$ |
|---|---|---|
| $f(x)$ | $0$ | |

## Problem Set 6.3

*In Problems 3–10, simplify the given expression.*

3. $e^{3 \ln x}$ $= \cancel{e^{\ln}} x^3 = x^3$

4. $e^{-2 \ln x}$ $= \cancel{e^{\ln}} x^{-2} = \frac{1}{x^2}$

5. $\ln e^{\cos x}$

6. $\cancel{\ln e}^{-2x-3}$ $= -2x-3$

7. $\ln(x^3 e^{-3x})$

8. $e^{x - \ln x}$

9. $e^{\ln 3 + 2 \ln x}$

10. $e^{\ln x^2 - y \ln x}$

*In Problems 11–22, find $D_x y$ (see Examples 1 and 2).*

11. $y = e^{x+2}$

12. $y = e^{2x^2 - x}$

5) $\ln e^{\cos x} = \cos x$

8) $e^{x - \ln x} = \frac{e^x}{\cancel{e^{\ln}} x} = \frac{e^x}{x}$

11) $D_x [e^{x+2}] = e^{x+2}$

13. $y = e^{\sqrt{x+2}}$

$D_x [e^{\sqrt{x+2}}] = \frac{1}{2\sqrt{x+2}} e^{\sqrt{x+2}} = \frac{e^{\sqrt{x+2}}}{2\sqrt{x+2}}$

15. $y = e^{2 \ln x}$ $= \cancel{e^{\ln}} x^2 = x^2$

$D_x = 2x$

$\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\sqrt{x^3} = \frac{3x^2}{2\sqrt{x^3}}$

$$\left(e^{x^2}\right)^1\left(e^{x^2}\right)^{-1/2}=\left(e^{x^2}\right)^{1/2}$$

**17.** $y = x^3 e^x$

$$D_x\left(x^3e^x\right)=x^3e^x+3x^2e^x=x^2e^x(x+3)$$

**19.** $y=\sqrt{e^{x^2}}+e^{\sqrt{x^2}}$ $=\left(e^{x^2}\right)^{1/2}+e^{\sqrt{x^2}}$

$$=\frac{1}{\cancel{2}}\left(e^{x^2}\right)^{-1/2}\cancel{2}x\,e^{x^2}+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2}}e^{\sqrt{x^2}}$$

$$x\left(e^{x^2}\right)^{1/2}+\frac{x}{|x|}e^{\sqrt{x^2}}$$

**21.** $e^{xy}+xy=2$ *Hint:* Use implicit differentiation.

$$e^{xy}(xy'+y)+(xy'+y)=0$$

$$xy'e^{xy}+ye^{xy}+xy'+y=0$$

$$xy'e^{xy}+xy'=-ye^{xy}-y$$

$$y'x\left(e^{xy}+1\right)=-y\left(e^{xy}+1\right)\qquad y'=\frac{-y\cancel{(e^{xy}+1)}}{x\cancel{(e^{xy}+1)}}=\frac{-y}{x}$$

*In Problems 25–36, first find the domain of the given function f and then find where it is increasing and decreasing, and also where it is concave upward and downward. Identify all extreme values and points of inflection. Then sketch the graph of $y = f(x)$.*

**27.** $f(x) = xe^{-x} \Rightarrow \frac{100}{e^{100}} \Rightarrow$

دائماً موجب

Domain $(-\infty, +\infty)$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$

$1 - x = 0 \qquad x = 1$

$+ \quad + \quad \underset{1}{|} \quad - \quad -$

increasing interval $(-\infty, 1]$

decreasing interval $[1, \infty]$

maximum $(1, \frac{1}{e})$


$f''(x) = e^{-x}(-1) + -e^{-x}(1-x)$

$= -e^{-x} - e^{-x} + xe^{-x}$

$f''(x) = e^{-x}(-1-1+x) = (x-2)e^{-x}$

$x - 2 = 0 \qquad x = 2$

$- \quad \underset{2}{|} \quad +$

concave up $(2, \infty)$ concave down $(-\infty, 2)$

inflection $(2, 2/e^2)$

**29.** $f(x) = \ln(x^2 + 1)$ Domian $(-\infty, \infty)$

$f'(x) = \dfrac{2x}{x^2+1}$ $2x = 0$ $x = 0$

- - - - | + + +
0

increasing $(-\infty, 0)$ minmum $(0, 0)$
decreasing $(0, \infty)$

$f''(x) = \dfrac{-2(x^2-1)}{(x^2+1)^2}$ $x^2 - 1 = 0$
$x^2 = 1$ $x = \pm 1$

- - - | + + + | - - -
-1 1

Concave up $(-1, 1)$

concave down $(-\infty, -1)$ $(1, \infty)$

inflection points $(-1, \ln 2)$
$(1, \ln 2)$

6.4

# General Exponential and Logarithmic Functions

exponential function to the base a

$e^x$ $a^x$ $5^x$

$\cancel{\ln e}^x = x$ $\cancel{e^{\ln}} x = x$

$\ln 3^x = x \ln 3$ $3^x = e^{\ln 3^x} = e^{x \ln 3}$

$$\boxed{a^x = e^{x \ln a} \qquad \ln a^x = x \ln a}$$

مشتقة وتكامل الداله الاصيه ذات الاساس (ثابت)

$$\boxed{D_x a^x = a^x \ln a}$$

$$\boxed{\int a^x \, dx = \frac{a^x}{\ln a} + C}$$

$$D_x\, 3^x = 3^x \ln 3$$

$$\int 3^x \, dx = \frac{3^x}{\ln 3} + C$$

$$D_x \sqrt{x} = \frac{1}{2\sqrt{x}}$$

**EXAMPLE 1** Find $D_x\left(3^{\sqrt{x}}\right)$.

$$D_x\, 3^{\sqrt{x}} = 3^{\sqrt{x}} \cdot \ln 3 \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{3^{\sqrt{x}} \ln 3}{2\sqrt{x}}$$

**EXAMPLE 2** Find $dy/dx$ if $y = (x^4 + 2)^5 + 5^{x^4+2}$.

**SOLUTION**

$$\frac{dy}{dx} = 5(x^4+2)^4 \cdot 4x^3 + 5^{x^4+2} \cdot \ln 5 \cdot (4x^3)$$

$$= 20x^3(x^4+2)^4 + 4x^3 \ln 5 \; 5^{x^4+2}$$

$$= 20x^3(x^4+2)^4 + 4x^3 \ln 5 \left[5^1 \cdot 5^{x^4+1}\right]$$

$$= 20x^3\left[(x^4+2)^4 + \ln 5 \left(5^{x^4+1}\right)\right]$$

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

$$\log_2 8 = \frac{\ln 8}{\ln 2}$$

$$\boxed{D_x \log_a x = \frac{1}{x \ln a}}$$

$$D_x \log_4 (5x^2+3x) = \frac{10x+3}{(5x^2+3x)\ln 4}$$

**EXAMPLE 4** If $y = \log_{10}(x^4 + 13)$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{4x^3}{(x^4+13)\ln 10}$$

**EXAMPLE 6** If $y = (x^2 + 1)^{\pi} + \pi^{\sin x}$, find $dy/dx$.

$$\frac{dy}{dx} = \pi(x^2+1)^{\pi-1} \cdot 2x + \pi^{\sin x} \cdot \ln\pi \cdot \cos x$$

**EXAMPLE 5** If $y = x^x$, $x > 0$, find $D_x y$ by two different methods.

طريقة ① ادخال ln للطرفين ثم اشتقاق ضمني

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{1}{y} y' = x \frac{1}{x} + \ln x$$

$$\frac{1}{y} y' = 1 + \ln x$$

$$y' = y(1 + \ln x)$$

$$\boxed{y' = x^x (1 + \ln x)}$$

طريقة ②

$$y = x^x = e^{\ln x^x} = e^{x \ln x}$$

$$y' = (x \frac{1}{x} + 1 \ln x) e^{x \ln x} = (1 + \ln x) \cancel{e^{x \ln x}}$$

$$\boxed{y' = (1 + \ln x) x^x}$$

**EXAMPLE 7** If $y = (x^2 + 1)^{\sin x}$, find $\frac{dy}{dx}$.

$$\ln y = \ln (x^2 + 1)^{\sin x}$$

$$\ln y = \sin x \ln(x^2 + 1)$$

$$\frac{1}{y} y' = \sin x \frac{2x}{x^2 + 1} + \cos x \ln(x^2 + 1)$$

$$y' = y\left[\frac{2x}{x^2+1}\sin x + \cos x \ln(x^2+1)\right]$$

$$y' = (x^2+1)^{\sin x}\left[\frac{2x}{x^2+1}\sin x + \cos x \ln(x^2+1)\right]$$

## Problem Set 6.4

*In Problems 17–26, find the indicated derivative or integral.*

**17.** $D_x(6^{2x})$

**18.** $D_x(3^{2x^2-3x})$

**19.** $D_x \log_3 e^x$

**20.** $D_x \log_{10}(x^3 + 9)$

**21.** $D_z[3^z \ln(z + 5)]$

**22.** $D_\theta \sqrt{\log_{10}(3^{\theta^2-\theta})}$

18) $D_x \left(3^{2x^2-3x}\right) = 3^{2x^2-3x} \cdot \ln 3 \cdot (4x-3)$

20) $D_x \log_{10}(x^3+9) = \dfrac{3x^2}{(x^3+9)\cdot \ln 10}$

22) $D_\theta \sqrt{\log_{10}(3^{\theta^2-\theta})}$

$D_\theta \sqrt{(\theta^2-\theta)\log_{10} 3}$

$D_\theta \left(\sqrt{\log_{10} 3}\,(\theta^2-\theta)^{1/2}\right)$

$\sqrt{\log_{10} 3}\; D_x\, (\theta^2-\theta)^{1/2}$

$\log_b a = \dfrac{\ln a}{\ln b}$

$\sqrt{\dfrac{\ln 3}{\ln 10}}\;\dfrac{1}{2}(\theta^2-\theta)^{-1/2}\cdot(2\theta-1) = \dfrac{\sqrt{\ln 3}\,(2\theta-1)}{2\sqrt{\ln 10}\sqrt{\theta^2-\theta}}$

*In Problems 27–32, find $dy/dx$. Note: You must distinguish among problems of the type $a^x$, $x^a$, and $x^x$ as in Examples 5–7.*

**27.** $y = 10^{(x^2)} + (x^2)^{10}$ **28.** $y = \sin^2 x + 2^{\sin x}$

**29.** $y = x^{\pi+1} + (\pi + 1)^x$ **30.** $y = 2^{(e^x)} + (2^e)^x$

**31.** $y = (x^2 + 1)^{\ln x}$ **32.** $y = (\ln x^2)^{2x+3}$

**33.** If $f(x) = x^{\sin x}$, find $f'(1)$.

28) $y = \sin^2 x + 2^{\sin x}$

$\frac{dy}{dx} = 2\sin x \cdot \cos x + 2^{\sin x} \ln 2 \cos x$

30) $y = 2^{(e^x)} + (2^e)^x$

$= 2^{(e^x)} \cdot \ln 2 \cdot e^x + (2^e)^x e \ln 2$

C **34.** Let $f(x) = \pi^x$ and $g(x) = x^\pi$. Which is larger, $f(e)$ or $g(e)$? $f'(e)$ or $g'(e)$?

$f(x) = \pi^x$ $g(x) = x^\pi$

$f(e) = \pi^e$ $<$ $g(e) = e^\pi$

22.46 23.14

$f'(x) = \pi^x \ln \pi$ $g'(x) = \pi x^{\pi-1}$

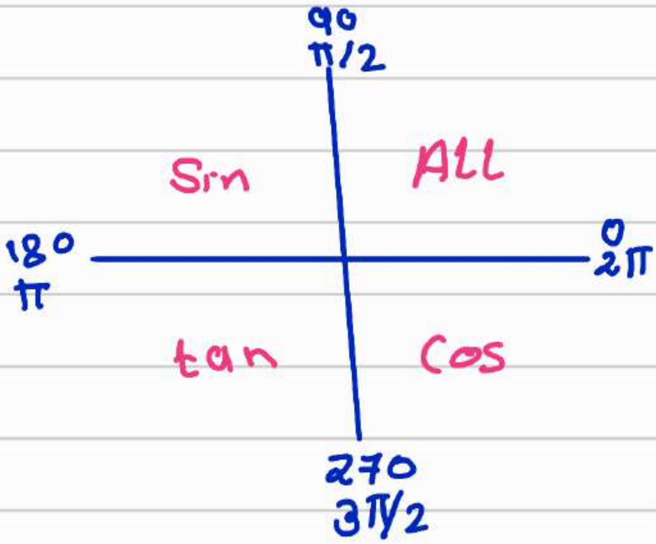
$f'(e) = \pi^e \ln \pi$ $g'(e) = \pi (e)^{\pi-1}$

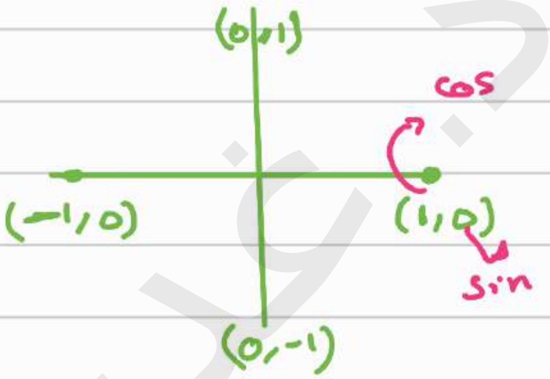
25.7 $<$ 26.7

# Trigonometric Functions

$\sin x$ $\qquad$ $\cos x$ $\qquad$ $\tan x = \dfrac{\sin x}{\cos x}$

$\csc x = \dfrac{1}{\sin x}$ $\qquad$ $\sec x = \dfrac{1}{\cos x}$ $\qquad$ $\cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$




$\cos(0) = 1$ $\qquad$ $\sin 0 = 0$ $\qquad$ $\tan 0 = 0$

$\cos\frac{\pi}{2} = 0$ $\qquad$ $\sin\frac{\pi}{2} = 1$ $\qquad$ $\tan 0 = \infty$

| | 0° | 30° | 45° | 60° | 90° | 120° | 180° | 270° | 360° |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $\pi$ | $3\pi/2$ | $2\pi$ |
| sin θ | 0 | $1/2$ | $1/\sqrt{2}$ | $\sqrt{3}/2$ | 1 | $\sqrt{3}/2$ | 0 | -1 | 0 |
| cos θ | 1 | $\sqrt{3}/2$ | $1/\sqrt{2}$ | $1/2$ | 0 | $-1/2$ | -1 | 0 | 1 |
| tan θ | 0 | $1/\sqrt{3}$ | 1 | $\sqrt{3}$ | ∞ | $-\sqrt{3}$ | 0 | ∞ | 0 |

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\sin^2 x + \cos^2 x = 1$

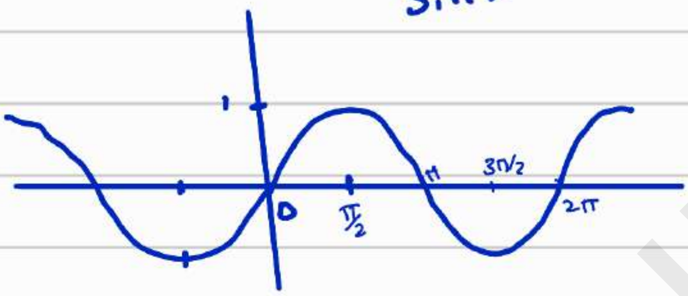
$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

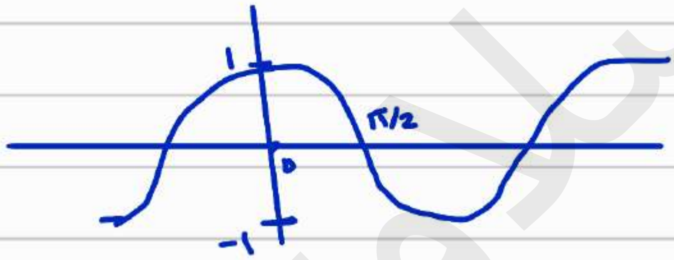
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin x$




$x \in [R)$

$y = [-1, 1]$




$x \in R$

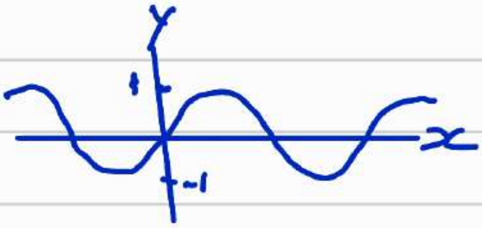
$y \in [-1, 1]$

$x = R - \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

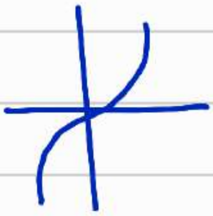
$y = R$

# 6.8 The Inverse Trigonometric Functions and Their Derivatives

$y = \sin x$



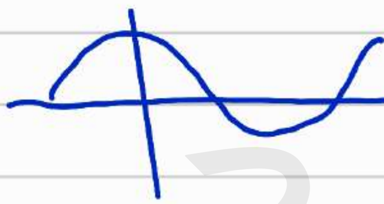

$y = \sin^{-1}(x)$



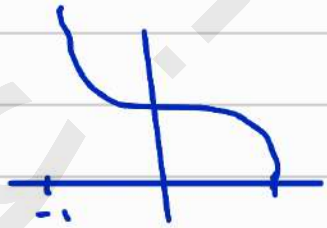
$x \in [-1, 1]$
$y = [-\frac{\pi}{2}, \frac{\pi}{2}]$
$-90 \quad 90$

$\sin^{-1}(0) = 0$

$y = \cos x$



$y = \cos^{-1}(x)$




$x \in [-1, 1]$
$y \in [0, \pi]$
$0 \quad 180$

$\cos^{-1}(1) = 0$

$y = \tan^{-1} x$

$x \in \mathbb{R}$
$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin^{-1}(-x) = -\sin^{-1}(x)$

$\tan^{-1}(-x) = -\tan^{-1}(x)$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

$\sin \sin^{-1} x = x$
$\sin^{-1} \sin x = x$

$\sec^{-1} x = \cos^{-1} \frac{1}{x}$
$\csc^{-1} x = \sin^{-1} \frac{1}{x}$

$\frac{2\pi}{3} \times \frac{180}{\cancel{\pi}} = 120$

**EXAMPLE 1** Calculate

(a) $\sin^{-1}(\sqrt{2}/2)$,

(b) $\cos^{-1}(-\frac{1}{2})$,

(c) $\cos(\cos^{-1} 0.6)$, and

(d) $\sin^{-1}(\sin 3\pi/2)$

a) $\frac{\pi}{4}$

b) $\pi - \cos^{-1}\frac{1}{2}$

$\frac{3}{3}\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

c) $0.6$

d) $\sin^{-1}\left(\sin \frac{3\pi}{2}\right)$

$\frac{3\pi}{2} = \frac{-\pi}{2}$

**EXAMPLE 3** Calculate

(a) $\tan^{-1}(1)$,

(b) $\tan^{-1}(-\sqrt{3})$,

(c) $\tan^{-1}(\tan 5.236)$,

(d) $\sec^{-1}(-1)$, $= \cos^{-1}(-1)$

(e) $\sec^{-1}(2)$, and $\cos^{-1}\frac{1}{2}$

(f) $\sec^{-1}(-1.32) = \sec^{-1}\frac{-1}{1.32}$

a) $\frac{\pi}{4}$

b) $\tan^{-1}(-\sqrt{3})$

$= -\tan^{-1}\sqrt{3}$

$= -\frac{\pi}{3}$

c) $\tan^{-1}(\tan 5.236)$

$\tan^{-1}(-1.732) = -\tan^{-1}(1.732)$

$-1.047$

d) $\cos^{-1}(-1) = \pi - \cos^{-1}(1) = \pi$

e) $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

f) $\sec^{-1}(-1.32) = \cos^{-1}\left(-\frac{1}{1.32}\right)$

$= \cos^{-1}(-0.7575)$

$= \pi - \cos^{-1}(0.7575)$

$2.43$

قواعد

**Theorem A**

(i) $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$

(ii) $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

(iii) $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$

(iv) $\tan(\sec^{-1} x) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -\sqrt{x^2 - 1}, & \text{if } x \leq -1 \end{cases}$

**EXAMPLE 4** Calculate $\sin\left[2\cos^{-1}\left(\frac{2}{3}\right)\right]$.

$\sin 2\theta = 2\sin\theta\cos\theta$

$2\sin\cos^{-1}\frac{2}{3}\ \cancel{\cos}\,\cancel{\cos^{-1}}\frac{2}{3}$

$2\sqrt{1-\left(\frac{2}{3}\right)^2}\cdot\frac{2}{3} = \frac{4}{3}\sqrt{\frac{9}{9}-\frac{4}{9}}$

$\frac{4}{3}\sqrt{\frac{5}{9}} = \frac{4\sqrt{5}}{9}$

مراجعه

$D_x \sin x = \cos x$ $\qquad$ $D_x \sec = \sec x \tan x$

$D_x \cos x = -\sin x$ $\qquad$ $D_x \csc = -\csc x \cot x$

$D_x \tan x = \sec^2 x$ $\qquad$ $D \cot = -\csc^2 x$

مشتقات الدوال العكسية

**Theorem B** **Derivatives of Four Inverse Trigonometric Functions**

(i) $D_x \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1$

(ii) $D_x \cos^{-1} x = -\dfrac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1$

(iii) $D_x \tan^{-1} x = \dfrac{1}{1 + x^2}$

(iv) $D_x \sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2 - 1}}, \qquad |x| > 1$

$$D_x \sin^{-1} 3x = \frac{3}{\sqrt{1-(3x)^2}}$$

$$D_x\, 3\tan^{-1} 4x = \frac{12}{1+(4x)^2}$$

**EXAMPLE 5** Find $D_x \sin^{-1}(3x - 1)$.

$$= \frac{3}{\sqrt{1-(3x-1)^2}}$$

$$= \frac{3}{\sqrt{1-[9x^2-6x+1]}} = \frac{3}{\sqrt{6x-9x^2}}$$

## Problem Set 6.8

*In Problems 1–10, find the exact value without using a calculator.*

**1.** $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

**2.** $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

**3.** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

**4.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

*In Problems 39–54, find $dy/dx$.*

**39.** $y = \ln(2 + \sin x)$

**41.** $y = \ln(\sec x + \tan x)$

39) $D_x\left[\ln(2+\sin x)\right] = \dfrac{\cos x}{2+\sin x}$

41) $D_x \ln\left[\sec x + \tan x\right] = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

**44.** $y = \arccos(e^x) = \cos^{-1}(e^x)$ $\rightarrow \cos^{-1} = -\dfrac{1}{\sqrt{1-x^2}}$

**46.** $y = e^x \arcsin x^2$

44) $D_x\left[\cos^{-1}(e^x)\right] = \dfrac{-e^x}{\sqrt{1-e^x}}$

46) $D_x\left[e^x \sin^{-1}x^2\right]$

$$= e^x \frac{2x}{\sqrt{1-x^4}} + e^x \sin^{-1}x^2$$

48. $y = \tan(\cos^{-1} x)$

50. $y = (\sec^{-1} x)^3$

48) $D_x\left[\tan(\cos^{-1}x)\right] = D_x\left[\dfrac{\sin(\cos^{-1}x)}{\cancel{\cos}\,(\cancel{\cos}^{-1}x)}\right]$

$$D_x\left[\frac{(1-x^2)^{\frac{1}{2}}}{x}\right] = \frac{x\left[-\frac{1}{2}(1-x^2)^{-\frac{1}{2}}2x\right] - (1-x^2)^{\frac{1}{2}}}{x^2}$$

$$= \frac{-x^2}{x^2\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{x^2}\,\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

$$= \frac{-x^2}{x^2\sqrt{1-x^2}} - \frac{1-x^2}{x^2\sqrt{1-x^2}}$$

$$\frac{-\cancel{x^2}-1+\cancel{x^2}}{x^2\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

50. $y = (\sec^{-1} x)^3$

$$D_x\left[(\sec^{-1}x)^3\right] = \frac{3(\sec^{-1}x)^2}{|x|\sqrt{x^2-1}}$$

52. $y = \sin^{-1}\left(\dfrac{1}{x^2 + 4}\right)$

$$\frac{1}{\sqrt{1 - \left(\frac{1}{x^2+4}\right)^2}} \cdot \frac{-2x}{(x^2-4)^2}$$

$$\frac{1}{\sqrt{1 - \frac{1}{(x^2+4)^2}}} \cdot \frac{-2x}{(x^2-4)^2}$$

$$\frac{-2x}{\sqrt{(x^2-4)^2 - 1}} \cdot \frac{1}{(x^2-4)}$$

$$\frac{-2x}{x^2-4\sqrt{x^4-8x^2+15}}$$

54. $y = x \operatorname{arcsec}(x^2 + 1)$

$$\frac{1}{|x|\sqrt{x^2-1}}$$

$$D_x\left[x \sec^{-1}(x^2+1)\right]$$

$$= x \, \frac{2x}{|x^2+1|\sqrt{(x^2+1)^2-1}} + \sec^{-1}(x^2+1)$$

$$\frac{2x^2}{(x^2+1)\sqrt{x^4+2x^2}} + \sec^{-1}(x^2+1)$$

## 6.9 The Hyperbolic Functions and Their Inverses

In both mathematics and science, certain combinations of $e^x$ and $e^{-x}$ occur so often that they are given special names.

**Definition** Hyperbolic Functions

The hyperbolic sine, hyperbolic cosine, and four related functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = e^x + e^{-x}$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$\frac{1}{4}\left[\cancel{e^{2x}} + 2e^x e^{-x} + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2e^x e^{-x} - \cancel{e^{-2x}}\right]$$

$$\frac{1}{4}[2+2] = 1$$

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

$$\cosh x = \sqrt{1 + \sinh^2 x}$$

**Theorem A** Derivatives of Hyperbolic Functions

$\cos \to -\sin$

$$D_x \sinh x = \cosh x \qquad D_x \cosh x = \sinh x$$

$$D_x \tanh x = \operatorname{sech}^2 x \qquad D_x \coth x = -\operatorname{csch}^2 x$$

$$D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x \qquad D_x \operatorname{csch} x = -\operatorname{csch} x \coth x$$

$\tan(\sin$

**EXAMPLE 1** Find $D_x \tanh(\sin x)$.

$$[\operatorname{sech}^2(\sin x)]\cos x$$

**EXAMPLE 2** Find $D_x \cosh^2(3x - 1)$.

$$6\cosh(3x-1) \cdot \sinh(3x-1)$$

**EXAMPLE 3** Find $\int \tanh x \, dx$.

$$\int \frac{\sinh x}{\cosh x}\, dx =$$

$$\int \cancel{\frac{\sinh x}{u}} \, \frac{du}{\cancel{\sinh x}} = \int \frac{1}{u} = \ln u$$

$$\ln(\cosh x)$$

## Inverse Hyperbolic functions

$$\sinh^{-1} x = y$$

$$\Downarrow$$

$$x = \sinh y$$

$$\sin(30) = \frac{1}{2}$$

$$\sin^{-1}\frac{1}{2} = 30$$

$$D_x \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$$

$$D_x \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \qquad x > 1$$

$$D_x \tanh^{-1} x = \frac{1}{1-x^2}, \qquad -1 < x < 1$$

$\cot^{-1} \frac{1}{x}$

$$D_x \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}}, \qquad 0 < x < 1$$

مشتقات hyperbolic inverse

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \ge 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad -1 < x < 1$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \qquad 0 < x \le 1$$

طرق لتمثيل hypolic باستخدام ln

**EXAMPLE 4** Show that $D_x \sinh^{-1} x = 1/\sqrt{x^2+1}$ by two different methods.

$y = \sinh^{-1} x$

$x = 2x + 4y$

$1 = 2 + 4\,D_y$

$x = \sinh y$

اشتق

$1 = \cosh y \; D_y$

طريقة 1

$D_y = \dfrac{1}{\cosh y} = \dfrac{1}{\sqrt{1+\sinh^2 y}} = \dfrac{1}{\sqrt{1+x^2}}$

طريقة ②

$$y = \sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$$

$$D_x\left(\ln\left(x+\sqrt{x^2+1}\right)\right) = \frac{1 + \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{\left(x+\sqrt{x^2+1}\right)}$$

$$= \frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$$

$$= \frac{\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}}{\cancel{x+\sqrt{x^2+1}}}$$

$$D_x y = \frac{1}{\sqrt{x^2+1}}$$

## Problem Set 6.9

$\sinh 2x = 2\sinh x \cosh x$

*In Problems 13–36, find $D_x y$.*

**13.** $y = \sinh^2 x$

$D_x(\sinh^2 x) = 2\sinh x \cosh x$

$\sinh 2x$

**15.** $y = 5\sinh^2 x$

$D_y = 5 \cdot 2\sinh x \cosh x = 5\sinh 2x$

**18.** $y = \sinh(x^2 + x)$

$D_y = \cosh(x^2+x) \cdot (2x+1)$

**20.** $y = \ln(\coth x)$

$$D_y = \frac{-\operatorname{csch}^2 x}{\coth x} = \frac{-1}{\sinh^2 x} \; \frac{\sinh x}{\cosh x}$$

$$= \frac{1}{\sinh x \cosh x}$$

**22.** $y = x^{-2} \sinh x$

$$Dy = x^{-2} \cosh x - 2x^{-3} \sinh x$$

**24.** $y = \sinh x \cosh 4x$

$$Dy = 4 \sinh x \sinh 4x + \cosh x \cosh 4x$$

$$\cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$$

**28.** $y = \cosh^{-1}(x^3)$

$$= \frac{3x^2}{\sqrt{x^6 + 1}}$$

**30.** $y = \coth^{-1}(x^5)$ $= \tanh^{-1} x^{-5}$

$$Dx = \frac{-5x^{-6}}{1 - x^{-10}}$$

$$\frac{-5x^{-6}}{1 - \frac{1}{x^{10}}} = \frac{-5x^{4}}{x^{10} - 1}$$

**32.** $y = x^2 \sinh^{-1}(x^5)$

$$Dy = \frac{5x^6}{\sqrt{x^{10}+1}} + 2x \sinh^{-1}(x^5)$$

**35.** $y = \tanh(\cot x)$

$$Dy = \operatorname{sech}^2(\cot x) \cdot (-\csc^2 x)$$