Electromagnetic Field: Phy321
Exercise Ch1: Vector Analysis

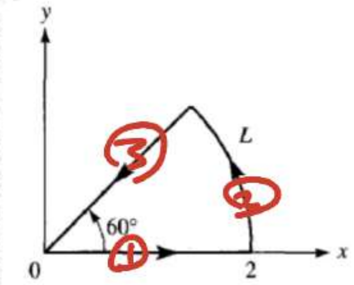
PRACTICE EXERCISE 3.2

Calculate the circulation of

$$\mathbf{A} = \rho \cos\phi \, \mathbf{a}_\rho + z \sin\phi \, \mathbf{a}_z$$

around the edge L of the wedge defined by $0 \le \rho \le 2$, $0 \le \phi \le 60^\circ$, $z = 0$ and shown in Figure 3.11.

Answer: 1.



$$dl = d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$$

$$\oint_L A\cdot dl = \int^{1} + \int^{2} + \int^{3} = 2 + 0 - 1 = \boxed{1}$$

$$^{1}\!\!\int = \int A\cdot dl = \int_0^2 \rho\cos\phi\,\hat{a}_\rho \cdot d\rho\,\hat{a}_\rho$$

$$= \int_0^2 \rho\cos\phi\, d\rho = \cos 0 \int_0^2 \rho\, d\rho = \left.\frac{\rho^2}{2}\right]_0^2$$

$$\boxed{^{1}\!\!\int = 2}$$

$$^{2}\!\!\int = \int A\cdot dl = \int z\sin\phi\, a_\phi \cdot \rho\, d\phi\, a_\phi$$

$$\boxed{^{2}\!\!\int = 0}$$

$$^{3}\!\!\int_2^0 \rho\cos\phi\, d\rho = \cos 60 \int_2^0 \rho\, d\rho = \frac{1}{2}\left[\frac{\rho^2}{2}\right]_2^0 = \boxed{-1}$$

PRACTICE EXERCISE 3.3

Determine the gradient of the following scalar fields:

(a) $U = x^2y + xyz$

(b) $V = \rho z \sin\phi + z^2 \cos^2\phi + \rho^2$

a) $\nabla U = \frac{\partial U}{\partial x}\hat{a}_x + \frac{\partial U}{\partial y}\hat{a}_y + \frac{\partial U}{\partial z}\hat{a}_z$

$\nabla U = (2xy + yz)a_x + (x^2 + xz)\hat{a}_y + (xy)\hat{a}_z$

b) $\nabla V = \frac{\partial V}{\partial \rho}a_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z$

تعديل

$= [z\sin\phi + 2\rho]\hat{a}_\rho + \frac{1}{\rho}[\rho z\cos\phi - z^2 2\cos\phi\sin\phi]\hat{a}_\phi$

$+ [\rho\sin\phi + 2z\cos^2\phi]a_z$

$= [z\sin\phi + 2\rho]\hat{a}_\rho + [z^2\cos\phi - \frac{z^2}{\rho}\sin 2\phi]a_\phi$

$+ [\rho\sin\phi + 2z\cos\phi]a_z$

PRACTICE EXERCISE 3.4

Given $\Phi = xy + yz + xz$, find gradient Φ at point (1, 2, 3) and the directional derivative of Φ at the same point in the direction toward point (3, 4, 4).

Answer: $5\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z$, 7.

①

$$\nabla \Phi = (y+z)a_x + (x+z)a_y + (y+x)a_z$$

at (1, 2, 3)

$$\boxed{\nabla \Phi = 5\hat{a}_x + 4\hat{a}_y + 3\hat{a}_z}$$

(5, 4, 3)

② Directional Derivative

A (1, 2 3) B(3, 4, 4)

Distance between A and B

$$\overline{AB} = (3-1, 4-2, 4-3)$$

$$= 2a_x + 2a_y + a_z$$

$$\hat{a} = \frac{2a_x + 2a_y + a_z}{\sqrt{2^2+2^2+1^2}} = \frac{2}{3}a_x + \frac{2}{3}a_y + \frac{1}{3}a_z$$

$\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$$\frac{d\Phi}{dt} = \nabla\Phi \cdot \vec{a} =$$

$$\frac{10}{3} + \frac{8}{3} + \frac{3}{3} = \frac{21}{3} = \boxed{7}$$

PRACTICE EXERCISE 3.6

Determine the divergence of the following vector fields and evaluate them at the specified points.

(a) $\mathbf{A} = yz\mathbf{a}_x + 4xy\mathbf{a}_y + y\mathbf{a}_z$ at $(1, -2, 3)$

(b) $\mathbf{B} = \rho z \sin\phi\, \mathbf{a}_\rho + 3\rho z^2 \cos\phi\, \mathbf{a}_\phi$ at $(5, \pi/2, 1)$

a) $\nabla\cdot A = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

$= 0 + 4x + 0 = 4x$

at $(1,-2,3)$ $\quad \nabla\cdot A = 4(1) = 4$

b) $\nabla\cdot B = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho B_\rho) + \dfrac{1}{\rho}\dfrac{\partial B_\phi}{\partial\phi} + \dfrac{\partial B_z}{\partial\rho}$

$\nabla\cdot B = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho^2 z\sin\phi) + \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi}(3\rho z^2\cos\phi)$

$= 2z\sin\phi - 3z^2\sin\phi$

$\nabla\cdot B = (2-3z)\, z\sin\phi$

at $(5,\frac{\pi}{2},1)$ $\quad \nabla\cdot B = (2-3(1))\sin\pi/2$

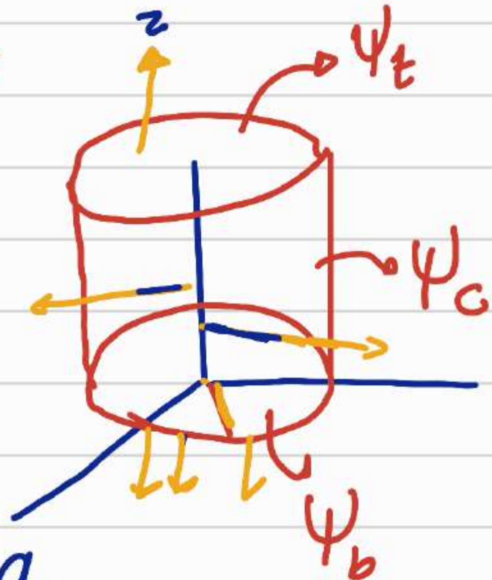
$\nabla\cdot B = -1$

PRACTICE EXERCISE 3.7

Determine the flux of $\mathbf{D} = \rho^2 \cos^2 \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_\phi$ over the closed surface of the cylinder $0 \le z \le 1$, $\rho = 4$. Verify the divergence theorem for this case.

Answer: 64π.

$$\Psi = \oint D \cdot ds = \int_v \nabla \cdot D \, dv$$



$$\Psi = \oint D \cdot ds = \Psi_t + \Psi_b + \Psi_c$$

Ψ_t $\quad z = 1 \quad ds = \rho \, d\rho \, d\phi \, a_z$

$$\Psi_t = \int (\rho^2 \cos^2 \phi \, a_\rho + z \sin \phi \, a_\phi) \cdot (\rho \, d\rho \, d\phi \, a_z)$$

$$\Psi_t = 0$$

Ψ_b $\quad z = 1 \quad \Psi_b = 0$

Ψ_c $\quad ds = \rho \, d\phi \, dz \, \underline{\hat{a}_\rho}$

$0 < z < 1 \quad \phi = 0 < \phi < 2\pi$

$$\psi_s = \int (\rho^2 \cos^2\phi \, \hat{a}_\rho) \cdot (\rho \, d\phi \, dz \, \hat{a}_\rho)$$

$$\psi_s = \int \rho^3 \cos^2\phi \quad d\phi \, dz$$

$$\psi_s = \rho^3 \underbrace{\int_0^{2\pi} \cos^2\phi \, d\phi}_{\pi} \int_0^1 dz$$

$$= 4^3 (\pi)(1) = \boxed{64\pi}$$

$$\psi = \int_v \nabla \cdot D \, dv$$

$$\nabla \cdot D = \frac{1}{\rho} \frac{\partial D}{\partial \rho} \rho^3 \cos^2\phi + \frac{1}{\rho} \frac{\partial}{\partial \phi} z \sin\phi$$

$$\nabla \cdot D = 3\rho \cos^2\phi + \frac{z}{\rho} \cos\phi$$

$$\psi = \int \left(3\rho \cos^2\phi + \frac{z}{\rho} \cos\phi\right) \rho \, d\rho \, d\phi \, dz$$

$$\psi = \int \underline{3\rho^2} \cos^2\phi \, \underline{d\rho} \, d\phi \, dz + z \cos\phi \, d\rho \, d\phi \, dz$$

$$\psi = 3\int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2\varphi\, d\varphi \int_0^1 dz + \int_0^1 z\,dz$$

$$\int_0^{2\pi} \cos\varphi\, d\varphi \int_0^4 d\rho$$

zero

$$= 3\left.\frac{\rho^3}{3}\right]_0^4 \left[\pi\right]\left[1\right] + \cancel{\left.\frac{z^2}{2}\right]_0^1 \left[\sin\varphi\right]_0^{2\pi}\left[4\right]}$$

$$3\frac{(64)}{3}\pi = \boxed{64\pi}$$

PRACTICE EXERCISE 3.8

Determine the curl of the vector fields in Practice Exercise 3.6 and evaluate them at the specified points.

a)

$$A = yz\,a_x + 4xy\,a_y + y\,a_z \qquad (1,-2,3)$$

$$\nabla \times A = \begin{vmatrix} \hat{a}_x & \hat{a}_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 4xy & y \end{vmatrix}$$

$$\left(\frac{\partial y}{\partial y} - \frac{\partial}{\partial z}4xy\right)a_x - \left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial z}yz\right)a_y + \left(\frac{\partial}{\partial x}4xy - \frac{\partial}{\partial y}yz\right)\hat{a}_z$$

$$(1-0)a_x - (0-y)\hat{a}_y + (4y-z)\hat{a}_z$$

$$\nabla \times A = \hat{a}_x + y\hat{a}_y + (4y-z)\hat{a}_z$$

at $(1,-2,3)$ $\boxed{\nabla \times A = a_x - 2a_y - 11\hat{a}_z}$

B) $B = \rho z \sin\phi\, \hat{a}_\rho + 3\rho z^2 \cos\phi\, \hat{a}_\phi$ $(5, \frac{\pi}{2}, 1)$

$$\nabla \times B = \frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho z \sin\phi & 3\rho^2 z^2 \cos\phi & 0 \end{vmatrix}$$

$$\nabla \times B = \frac{1}{\rho}\Big\{\left(\frac{\partial}{\partial \phi}0 - \frac{\partial}{\partial z}3\rho^2 z^2\cos\phi\right)\hat{a}_\rho - \left(\frac{\partial}{\partial \rho}0 - \frac{\partial}{\partial z}\rho z \sin\phi\right)\rho a_\phi + \left(\frac{\partial}{\partial \rho}3\rho^2 z^2 \cos\phi - \frac{\partial}{\partial \phi}\rho z \sin\phi\right)a_z\Big\}$$

$$\nabla \times B = \frac{1}{\rho}\Big\{-6\rho^2 z\cos\phi\,\hat{a}_\rho + \rho^2 \sin\phi\,\hat{a}_\phi$$
$$(6\rho z^2\cos\phi + \rho z\cos\phi)\hat{a}_z\Big\}$$

$$\nabla \times B = -6\rho z \underbrace{\cos\phi}_{0}\, \hat{a}_\rho + \rho \sin\phi\, \hat{a}_\phi$$
$$+ (6z-1)\, z \underbrace{\cos}_{0}\phi\, \hat{a}_z$$

$$\nabla \times B = 5\left(\sin \tfrac{\pi}{2}\right) \hat{a}_\phi = 5\hat{a}_\rho$$

PRACTICE EXERCISE 3.9

Use Stokes's theorem to confirm your result in Practice Exercise 3.2.

$$\oint A \cdot dl = \int_s \nabla \times A \cdot ds$$

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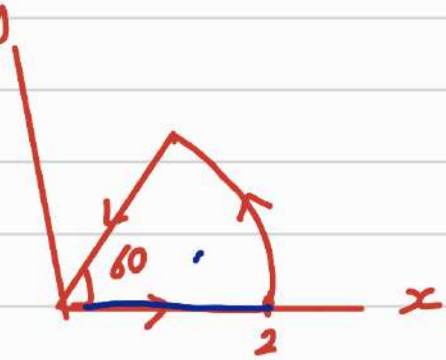

$$\nabla \times A = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho\cos\phi & \rho(0) & z\sin\phi \end{vmatrix}$$

$$\nabla \times A = \frac{1}{\rho}\left[z\cos\phi - 0\right]\hat{a}_\rho - \frac{1}{\rho}\left[0 - 0\right]\rho\,\hat{a}_\phi$$

$$\frac{1}{\rho}\left[0 - -\rho\sin\phi\right]a_z$$

$$\nabla \times A = \frac{1}{\rho}\left[z\cos\phi\, a_\rho\right] + \left[\sin\phi\, a_z\right]$$

$a_z \cdot a_z = 1$

$$\oint (\nabla \times A) \cdot ds = \int_s \left[\frac{z}{\rho}\cos\phi\,\hat{a}_\rho + \sin\phi\, a_z\right] \cdot \left[\rho\, d\rho\, d\phi\, \hat{a}_z\right]$$

zero

$$\int (\nabla \times A)\cdot ds = \int_s \rho \sin\phi \, d\rho \, d\phi$$

$$= \int_0^2 \rho \, d\rho \int_0^{60} \sin\phi \, d\phi$$

$$= \frac{\rho^2}{2}\Big]_0^2 \quad -\cos\phi\Big]_0^{60}$$

$$\left(\frac{4-0}{2}\right)\left(-\cos 60 - \cos 0\right)$$

$$2 \quad \left(-\frac{1}{2}+1\right)$$

$$2\left(\frac{1}{2}\right) = 1$$

PRACTICE EXERCISE 3.10

For a scalar field V, show that $\nabla \times \nabla V = 0$; that is, the curl of the gradient of any scalar field vanishes.

(A_x, A_y, A_z)

Answer: Proof.

$$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$$

$$\nabla \times \nabla V = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right]\hat{a}_x - \left[\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial x \partial z}\right]\hat{a}_y$$

$$+ \left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right]$$

$$0\,\hat{a}_x + 0\,a_y + 0\,a_z$$

$$= 0$$