Signals & Systems

1. Signals and Systems

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September 12, 2025

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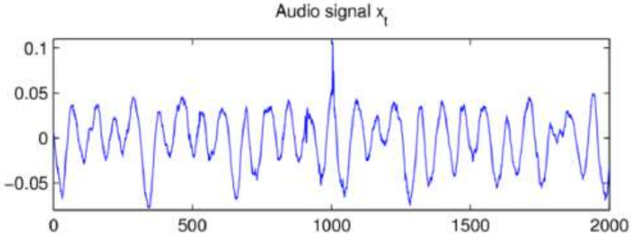
Why Study Signals and Systems?

- Helps design reliable and safe engineering systems.
- Enables mathematical modeling and simulation.
- Prevents failures – performance, cost, safety.

Signals

What is a Signal?



What is a Signal?

A **signal** is a **function** that conveys information about a physical phenomenon (information) about how something changes over time or space.

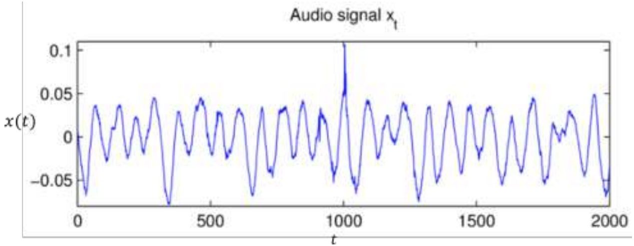
Examples:

- My voice in this lecture is a signal because the sound waves change over time
- A video is a signal because the brightness, colors, and the frames change over time
- Voltage $v(t)$ or current $i(t)$ in an electronic circuit
- The position, velocity, or acceleration of an object
- Digital image, digital video, or digital audio
- The temperature of a room over a day is a signal
- Stock index over a day is a signal

What is a Signal?

Mathematically, a signal is represented as a function of an independent variable t. Usually t represents time. Thus, a signal is denoted by $x(t)$.

- A **signal** maps an independent variable to a value: x : domain $\rightarrow \mathbb{R}$ (or $\mathbb{C}$).



A signal $x(t_1, t_2, \ldots, t_n)$:

- t_k = **independent variables**
- x = **dependent variable**

Signal Dimensionality

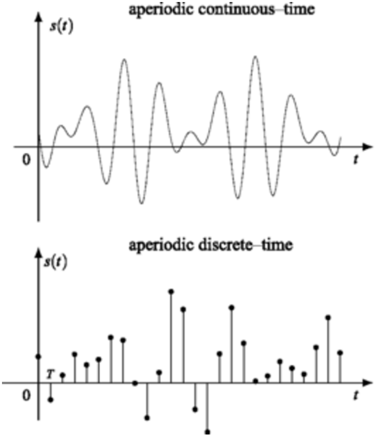
Number of independent variables (i.e., dimensionality):

- **1D Signal:** One independent variable (e.g., time)
  - Audio signal (pressure as a function of time)
- **Multi-D Signal:** More than one variable
  - Image (light intensity over x and y)
  - 2D: Image - intensity over (x, y)
  - 3D: MRI scan - intensity over (x, y, z)

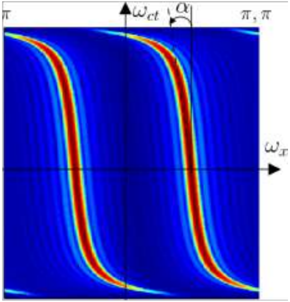
In this course

We will only deal with **one dimensional** signals

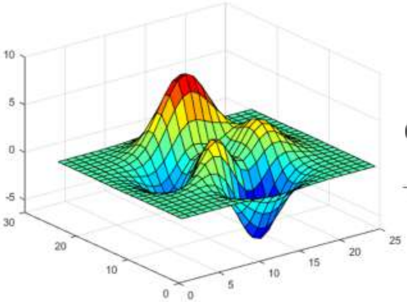
Examples of 1-D Signal

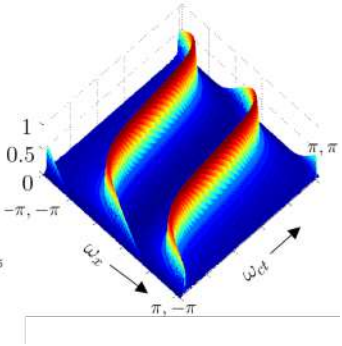


Examples of 2-D Signal



Examples of 3-D Signal

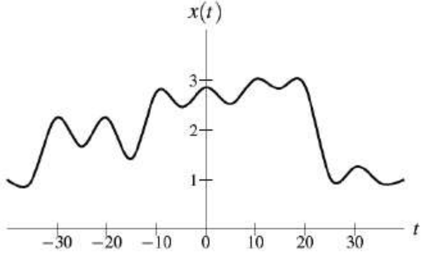




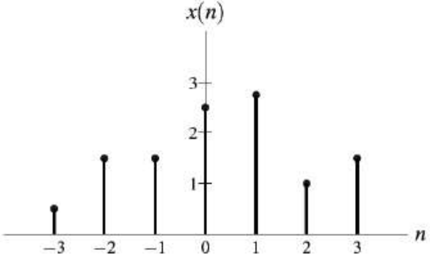
Continuous-Time (CT) vs Discrete-Time (DT)

- A signal with continuous independent variables is said to be **Continuous-Time (CT)**:
  - Independent variable is *continuous*
  - Defined at every instant in time
  - Example: Analog voltage waveform
- A signal with discrete independent variables is said to be **Discrete-Time (DT)**:
  - Independent variable is *discrete*
  - Defined only at specific time instants
  - Example: Daily temperature samples

Continuous-Time (CT) vs Discrete-Time (DT)



Continuous-Time (CT) Signal



Discrete-Time (DT) Signal

Continuous-valued (CV) vs. Discrete-valued (DV)

This classification is about amplitude (range), not the time axis.

- A signal with continuous independent variables is said to be **Continuous-valued (CV)**:
  - Values can be any real number in an interval (uncountably many).

$$x(t) \in \mathbb{R} \quad \text{or} \quad x[n] \in \mathbb{R}$$

  - Take any value in an interval
  - Example: Analog voltage waveform

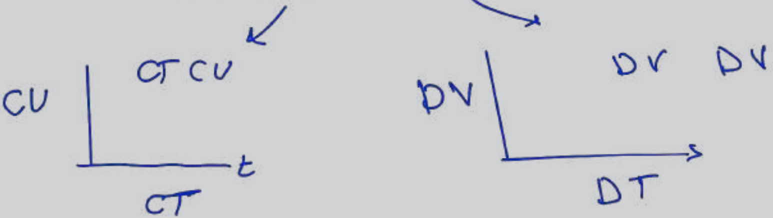
- A signal with discrete independent variables is said to be **Discrete-valued (DV)**:
  - Values come from a finite (or countable) set of levels.

$$x(t) \in \{\alpha_1, \ldots, \alpha_M\} \text{ or } x[n] \in \{\alpha_1, \ldots, \alpha_M\}$$

  - Takes only specific values (e.g., integers)
  - Example: Binary signal $\{0, 1\}$ or NRZ waveform $\{\pm 1\}$.

Orthogonality: You can have CT-CV (analog waveform), CT-DV (continuous time, finite levels–e.g. ideal square wave ± 1), DT-CV (sampled but unquantized), and DT-DV (digital sequence).

Analog vs Digital Signals



Analog vs Digital Signals

Analog Signal

A continuous-valued (CV) continuous-time (CT) signal is said to be analog.

- e.g., microphone output or voltage waveform.

Digital Signal

A discrete-valued (DV) discrete-time (DT) signal is said to be digital.

- e.g., binary data, digital audio in a computer

In this course

We will use

- $x(t)$ to denote continuous-time signals $x(t)$
- $x[n]$ or $\{x_n\}$ to denote discrete-time signals $x[n]$

Discrete-time signal = sequence

- A DT signal is a **sequence of numbers**: $\{x_n\}$ or $x[n]$, $n \in \mathbb{Z}$.
- Other times DT signals come from **sampling** a CT signal.

$$\{x_n\} = \{\ldots, x[-1], x[0], x[1], \ldots\}$$

- Sample a CT signal $x(t)$ at times $t_n = nT_s$ ($T_s > 0$ is the **sampling interval**).
- The DT sequence is

$$\boxed{x[n] = x_n = x(t_n) = x(nT_s)}, \quad t_n = nT_s, \qquad n \in \mathbb{Z}.$$

- Indices are integers: $n = \ldots, -2, -1, 0, 1, 2, \ldots$
- Sampling frequency: $f_s = \dfrac{1}{T_s}$ (samples per second).

T الفترة

التردد = 1 / الفترة (زمن)

Use brackets [ ] *for DT:* $x[n]$

and parentheses ( ) *for CT:* $x(t)$.

Defining a discrete-time signal

Two equivalent ways:

1. **By a rule (formula)** for the n-th value, e.g.

   معادلة

$$x[n] = \begin{cases} (1/2)^n, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

2. **By an explicit list** of values (a sequence):

$$\{x_n\} = \{\ldots, 0, 0, 1, 2, 2, \underset{\uparrow}{1}, 0, 1, 0, 2, 0, 0, \ldots\}$$

$$\{x_n\} = \{\ldots, 0, 0, 1, 2, 2, \underset{n=0}{1}, 0, 1, 0, 2, 0, 0, \ldots\}$$

(Convention: if no arrow/index is shown, the first listed element corresponds to $n = 0$ or $\uparrow$ and all earlier terms are 0.)

Four Signal Types = Time Axis $\times$ Amplitude

	Continuous-Valued (CV)	Discrete-Valued (DV)
Continuous-Time (CT)	**CT-CV** ■ *Analog waveform; defined for all t; amplitude real-valued.* ■ *Example:* $v(t) = 3\cos(2\pi f_0 t + \varphi)$	**CT-DV** ■ *Continuous time, but amplitudes from a finite set (levels).* ■ *Example:* $x(t) = \text{sgn}(\sin\omega_0 t) \in \{-1, +1\}$
Discrete-Time (DT)	**DT-CV** ■ *Samples at* $t = nT_s$*; each sample is real (unquantized).* ■ *Example:* $x[n] = e^{-0.1n}\sin(0.4\pi n)$	**DT-DV** ■ *Samples at* $t = nT_s$*; each sample from a finite set.* ■ *Example:* $x[n] \in \{0, 1\}$ (binary) or M-level PCM

How signals move between boxes: Sampling $\Rightarrow$ CT $\rightarrow$ DT; Quantization $\Rightarrow$ CV $\rightarrow$ DV.

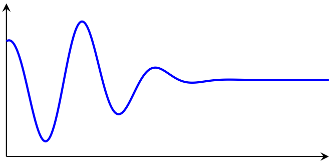
Four Signal Types: Time axis × Amplitude

Quantization: CV → DV

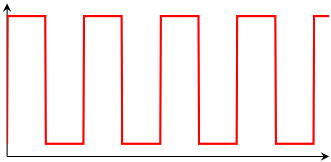
Sampling: CT → DT

	Continuous-Valued (CV)	**Discrete-Valued (DV)**
Continuous-Time (CT)	**CT-CV** (analog)	**CT-DV** (level/square)
Discrete-Time (DT)	**DT-CV** (samples, real-valued)	**DT-DV** (digital sequence)

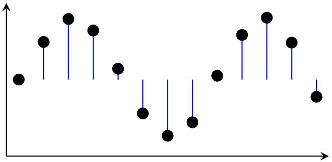
Four Signal Types (Time axis × Amplitude)



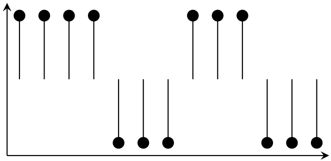
CT-CV: analog (continuous time, real amplitude)



CT-DV: level/square (continuous time, quantized amplitude)



DT-CV: samples, real-valued (discrete time, real amplitude)



DT-DV: digital sequence (discrete time, quantized amplitude)

Deterministic vs Random Signals

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Deterministic vs Random Signals

Deterministic

Values are **completely specified** for every time index.

Example (CT): $x(t) = A\cos(\omega_0 t + \varphi) \quad \Rightarrow \quad$ exactly predictable at all t.

Random (stochastic)

Values are **not predictable pointwise**; must be described *statistically* (e.g., mean, variance, distribution, autocorrelation).

Example (DT): $x[n] \sim$ i.i.d. noise, $\quad \mathbb{E}\{x[n]\} = 0, \; \mathrm{var}\{x[n]\} = \sigma^2.$

Random signals are usually treated via **probability & statistics**.

In this course

We will focuses on deterministic signals.

Signal Operations

Transformations of the Independent Variable

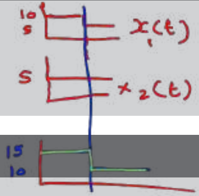
- Signal transformations are crucial for:
  - Signal analysis
  - System response computation
- Common transformations:
  - Time Shifting
  - Time Reversal
  - Time Scaling

Addition of CT Signals

Definition (pointwise)

Given $x_1(t), x_2(t) : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$),

$$\boxed{y(t) = x_1(t) + x_2(t)} \quad \text{for every } t.$$



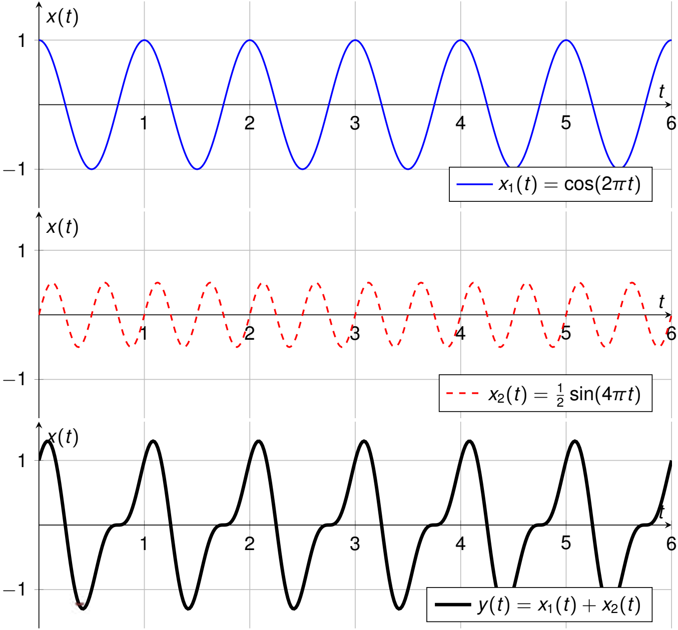
Key properties

- Commutative: $x_1 + x_2 = x_2 + x_1$
- Associative: $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$
- Identity & inverse: $x + 0 = x$, $x + (-x) = 0$

Example:

$$x_1(t) = \cos(2\pi t), \qquad x_2(t) = \tfrac{1}{2}\sin(4\pi t), \qquad y(t) = x_1(t) + x_2(t).$$

Addition of CT Signals

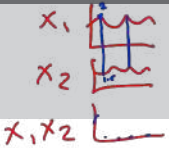


Multiplication of Continuous-Time Signals

Pointwise product

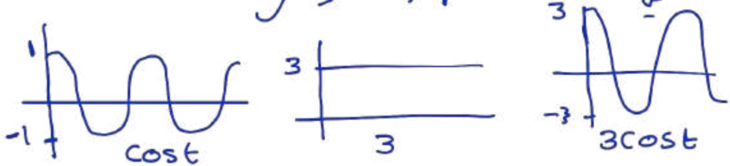
Given CT signals $x_1(t)$ and $x_2(t)$, their product is

$$\boxed{y(t) = x_1(t)\,x_2(t)}.$$

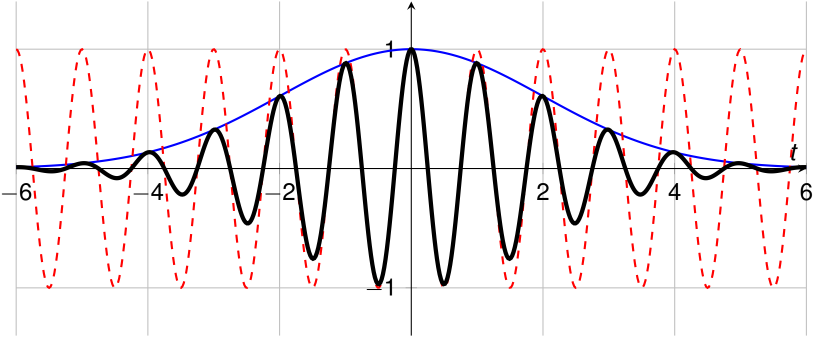


- Computed *at each instant t* (sample-by-sample).
- Zeros propagate: if $x_1(t_0) = 0$ or $x_2(t_0) = 0$, then $y(t_0) = 0$.
- If $x_2(t) = a$ (constant), then $y(t) = a\,x_1(t)$ (vertical scaling).

$$y = x_1 \; x_2$$
$$y = a x_1$$



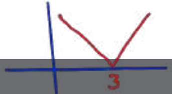
Multiplication (CT): Example



$x_1(t) = e^{-t^2/8}$ $x_2(t) = \cos(2\pi t)$ $y(t) = x_1(t)\, x_2(t)$

Step 1: show $x_1(t)$ (a time window). Step 2: add $x_2(t)$ (a cosine carrier). Step 3: product $y(t)$ = windowed cosine (amplitude-modulated).

Time Shifting (translation) of CT Signals



Definition

Time shifting maps the input signal $x(t)$ to the output

$$\boxed{y(t) = x(t - b)}, \qquad b \in \mathbb{R}.$$

x(t)

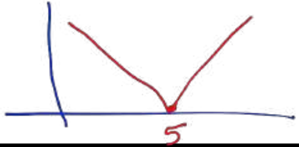
x(t+2)

ازاحه لليسار بخطوتين (سابقه)

- If $b > 0$: shift right (delay by b).
- If $b < 0$: shift left (advance by $|b|$).
- Shape is unchanged; only the time axis is displaced.

x(t − 2)

ازاحه لليمين خطوتين (تأخير)

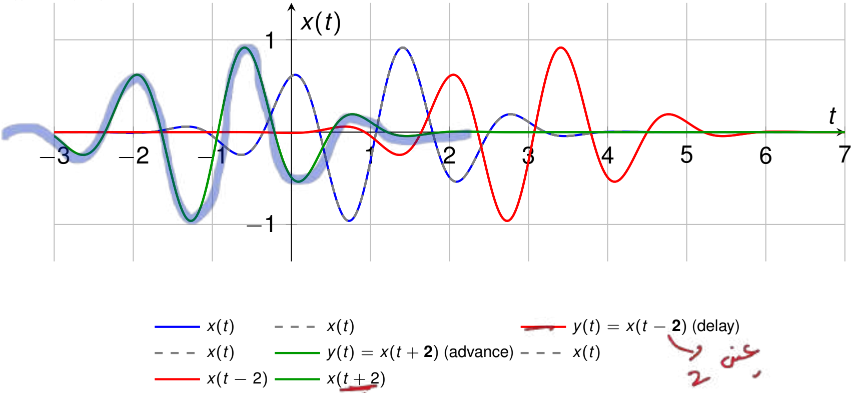


Time Shifting of CT Signals

Example used on the plot:

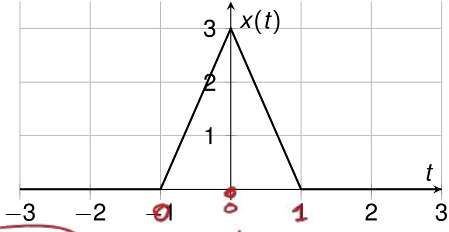
$$x(t) = e^{-\frac{1}{2}(t-1)^2} \cos(2\pi \cdot 0.7\, t).$$

legend columns=3, legend cell align=left, legend style=at=(0.5,-0.18), anchor=north, draw=none

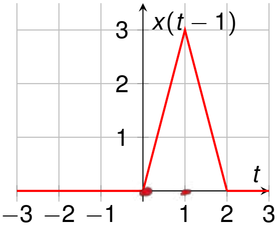


Time Shifting (CT Signals)

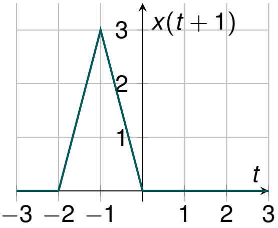
Original signal $x(t)$



Delayed by 1: $x(t-1)$



Advanced by 1: $x(t+1)$



Time Shifting of DT Signals

- **Time shifting** (translation) maps the input sequence $x[n]$ to the output

$$\boxed{y[n] = x[n-b]}$$

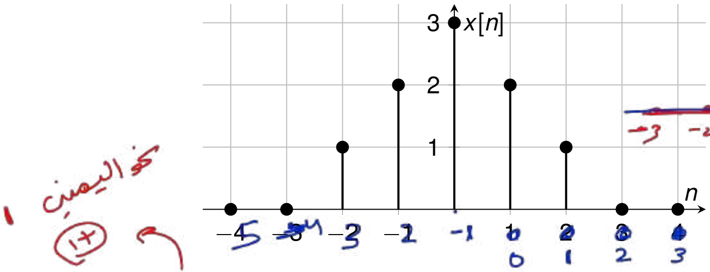
where $b \in \mathbb{Z}$ (an integer shift).

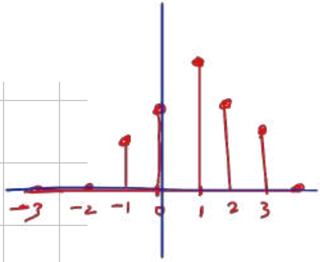
- The operation moves samples along the time (index) axis:
  - If $b > 0$: shift to the right (delay by b samples).
  - If $b < 0$: shift to the left (advance by $|b|$ samples).

b	$y[n]$	Effect
1	$x[n-1]$	delay by 1 sample (right shift)
2	$x[n-2]$	delay by 2 samples
-1	$x[n+1]$	advance by 1 sample (left shift)
-3	$x[n+3]$	advance by 3 samples

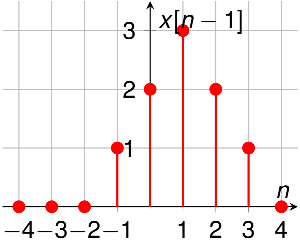
Time Shifting (DT Signal)

Original sequence $x[n]$

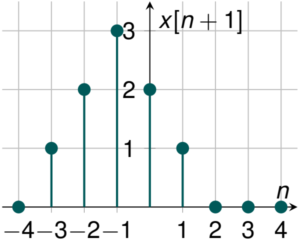




Delayed by 1: $x[n-1]$



Advanced by 1: $x[n+1]$

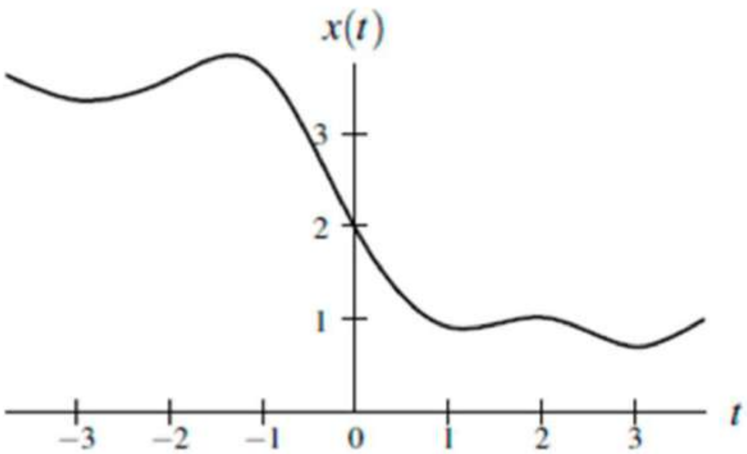


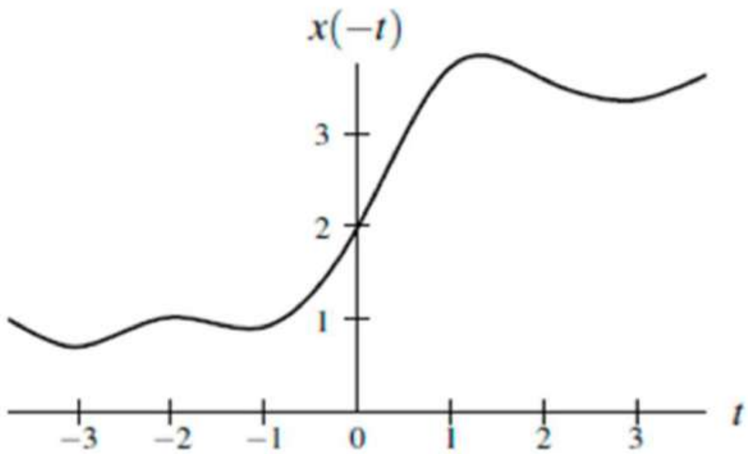
Time Reversal

- Time reversal (also known as reflection) maps the input signal x to the output signal y as given by

$$y(t) = x(-t).$$

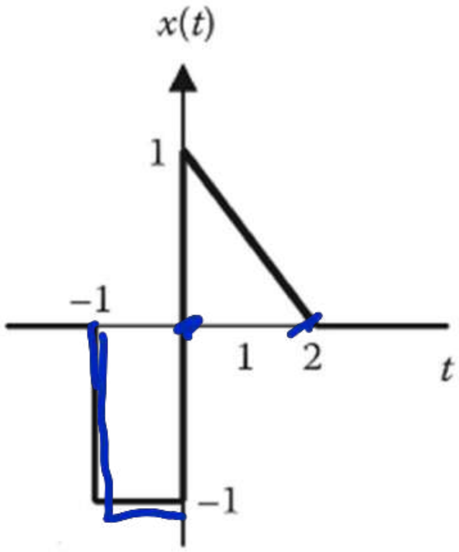
- Geometrically, the output signal y is a reflection of the input signal x about the vertical line $t = 0$.

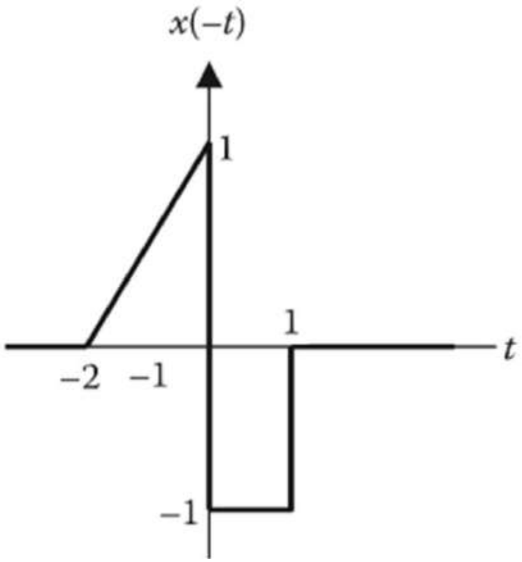


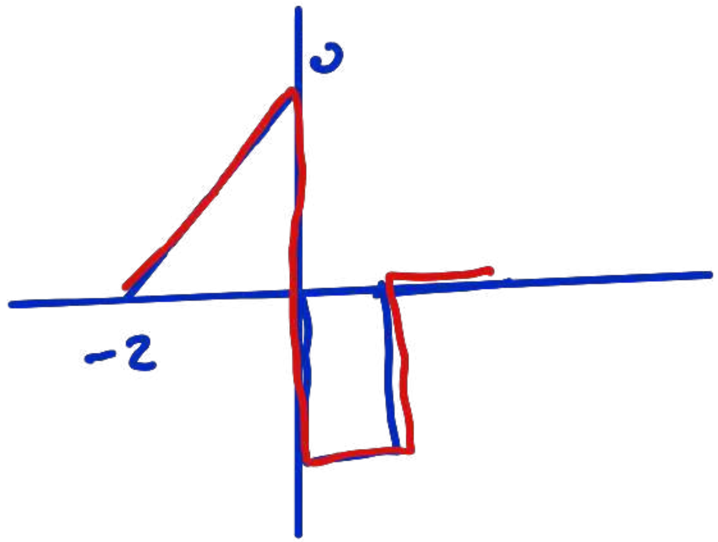


Time Reversal: Examples

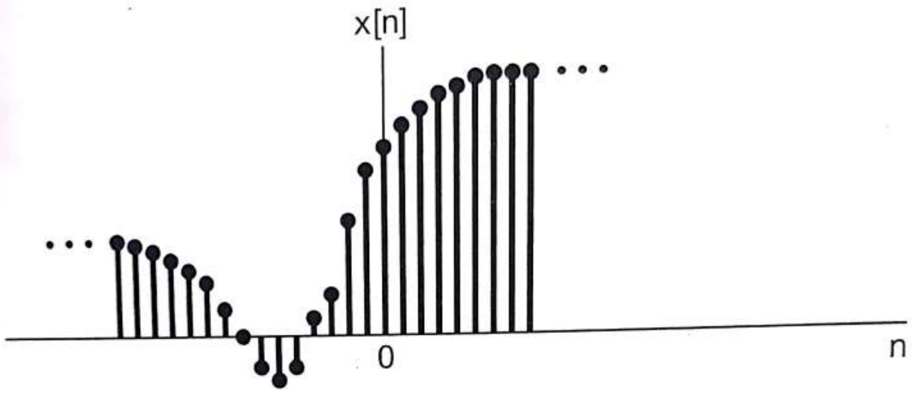
- For example, if $x(t)$ represents an audio signal, then $x(-t)$ is the same audio signal played backward
- Sketch $x(-t)$ for the following signal



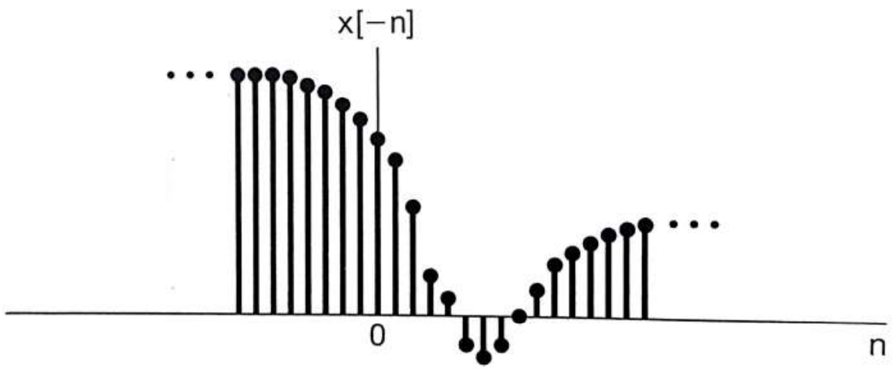




Time Reversal: DT Example



(a)

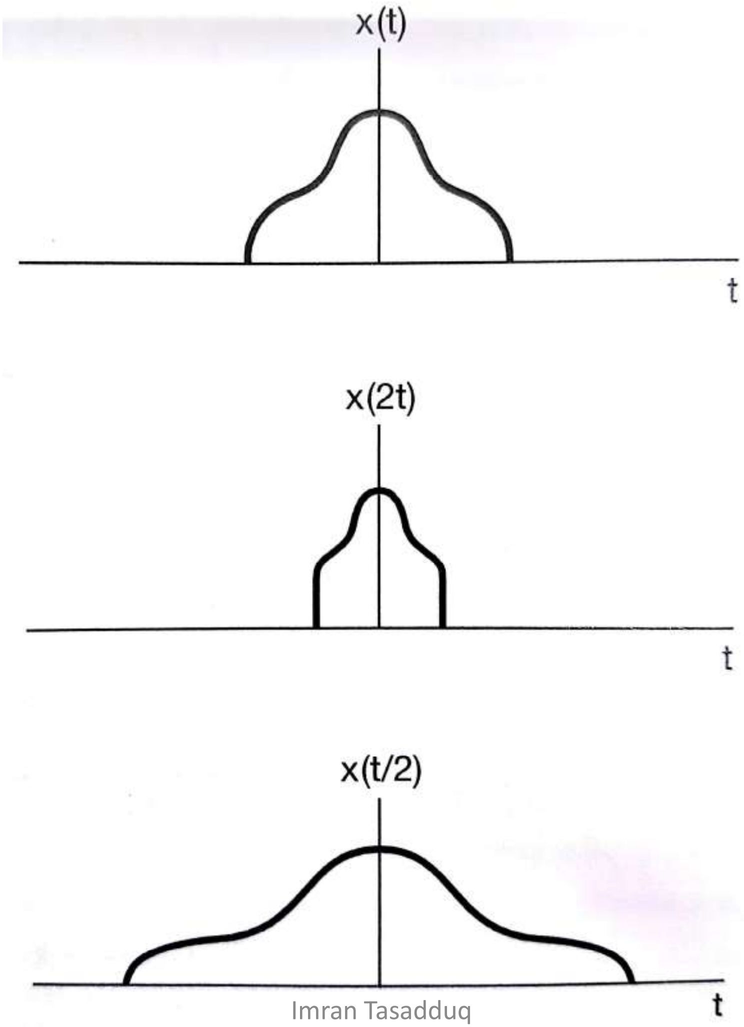


(b)

$x[n]$

$y = x[-n]$

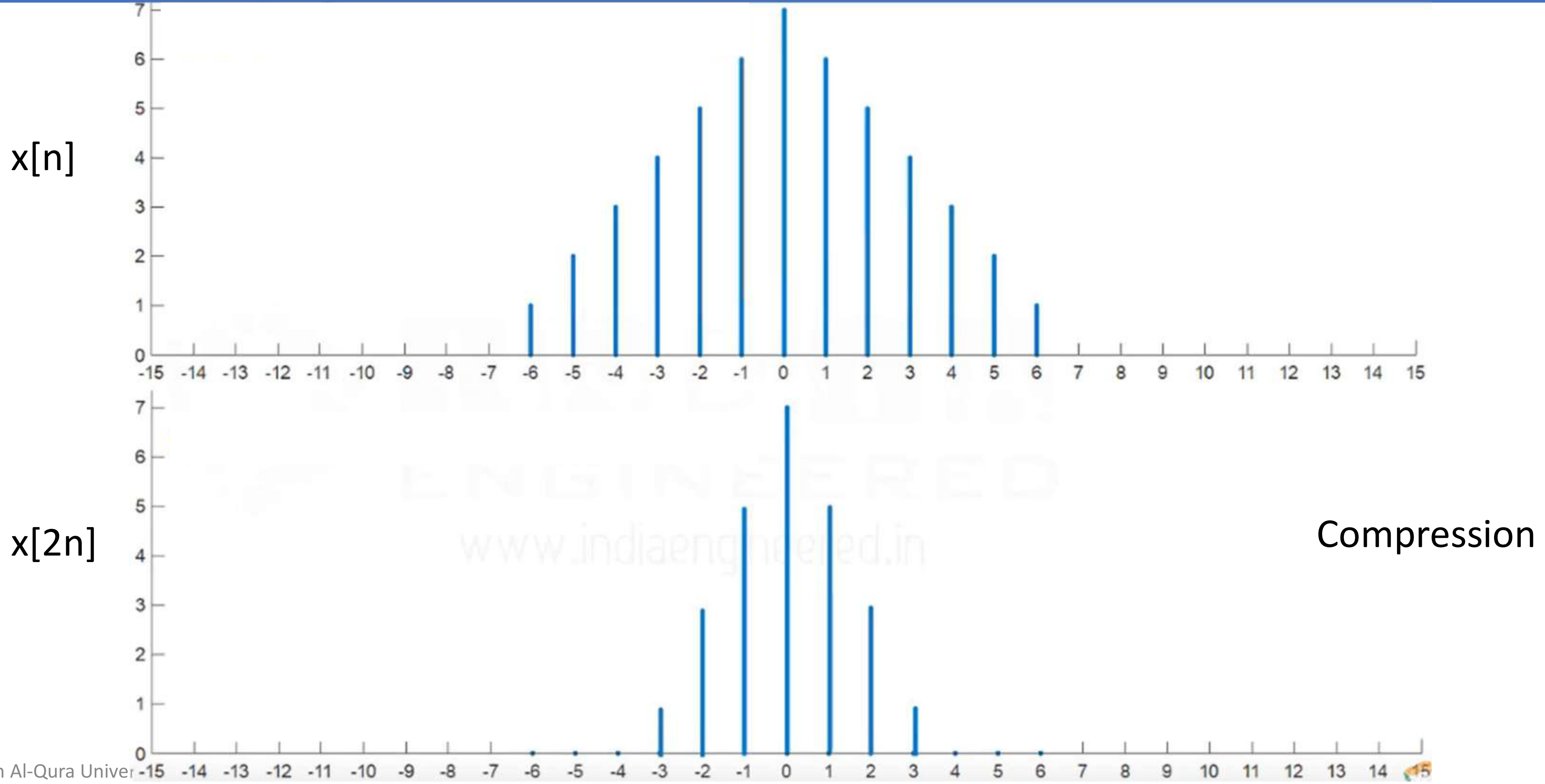
Time Scaling



Compression

Expansion

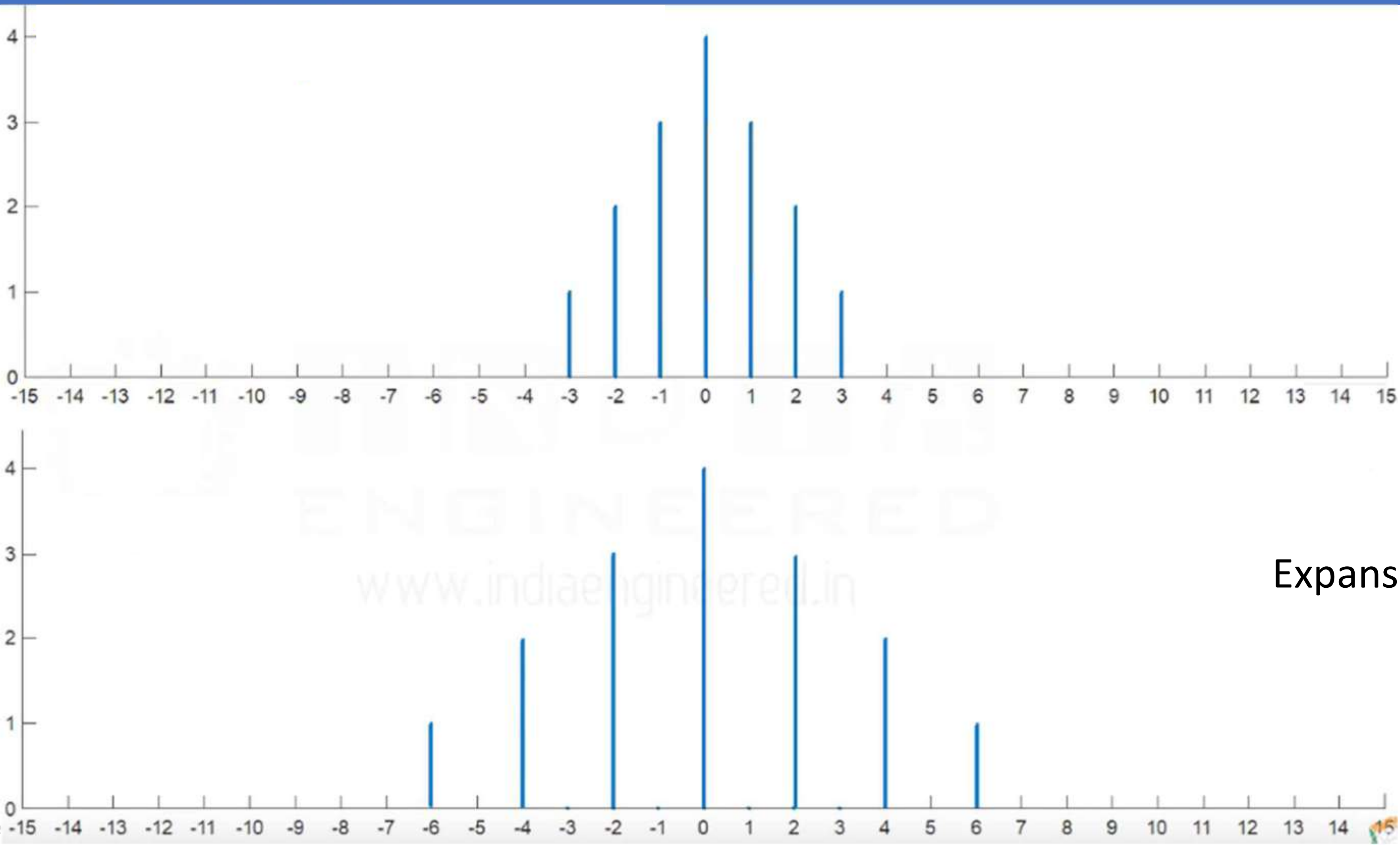
Time Scaling (DT Signals)



Time Scaling (DT Signals)

x[n]

x[n/2]



Expansion

Time Scaling

- Time scaling maps the input signal x to the output signal y as given by

$$y(t) = x(at),$$

where a is a nonzero real number.

- If $|a| > 1$, the signal is compressed
- If $|a| < 1$, the signal is expanded
- If $|a| = 1$, the signal is neither expanded nor compressed

$x(t)$

$y = x(at)$

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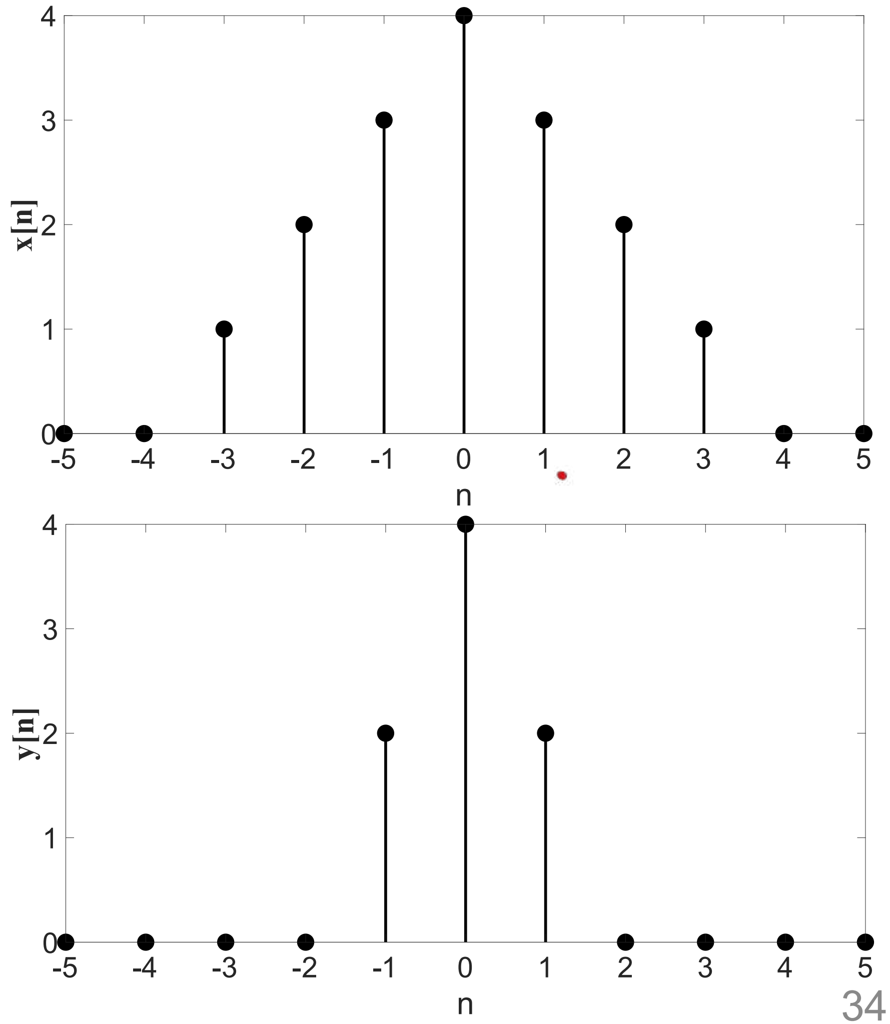
اقل من واحد تمدد

Time Scaling (Example of a DT Signal)

- Given a DT signal $x[n]$, sketch the signal $y[n]$ such that $y[n] = x[2n]$

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
x	0	0	1	2	3	4	3	2	1	0	0
y	0	0	0	0	2	4	2	0	0	0	0

- $y[0] = x[2 \times 0] = x[0] = 4$
- $y[1] = x[2 \times 1] = x[2] = 2$
- $y[2] = x[2 \times 2] = x[4] = 0$
- $y[-1] = x[2 \times -1] = x[-2] = 2$
- $y[-2] = x[2 \times -2] = x[-4] = 0$



- Given a DT signal $x[n]$, sketch the signal $y[n]$ such that $y[n] = x[2n]$

$2 > 1$ انضغاط
تضغيط، $\frac{1}{2}$

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
x	0	0	1	2	3	4	3	2	1	0	0
y				0	2	4	2	0			

$y[n] = x[2n]$

$n = 0 \quad y[0] = x[0]$

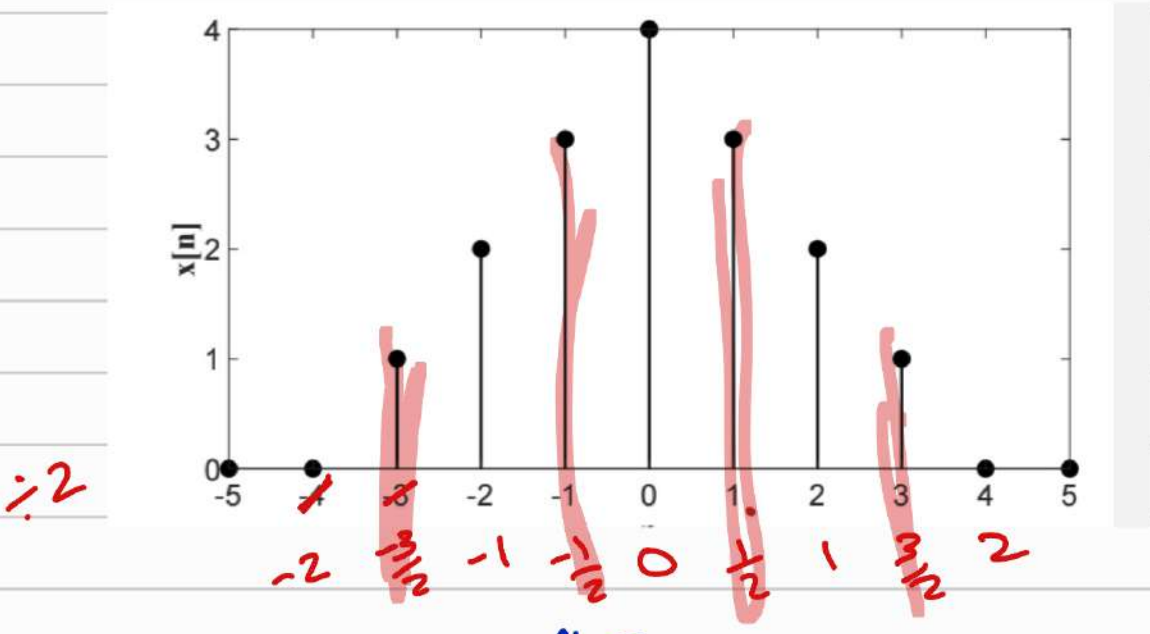
$n = 1 \quad y[1] = x[2]$

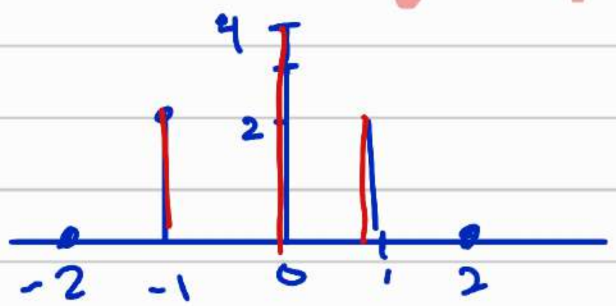
$n = 2 \quad y[2] = x[4]$

$n = -1$

$y[-1] = x[-2]$

$y[-2] = x[-4]$



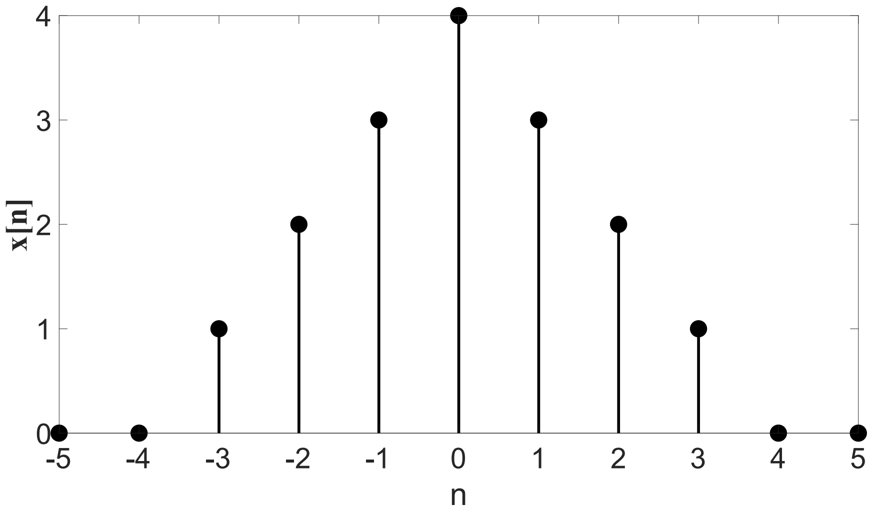


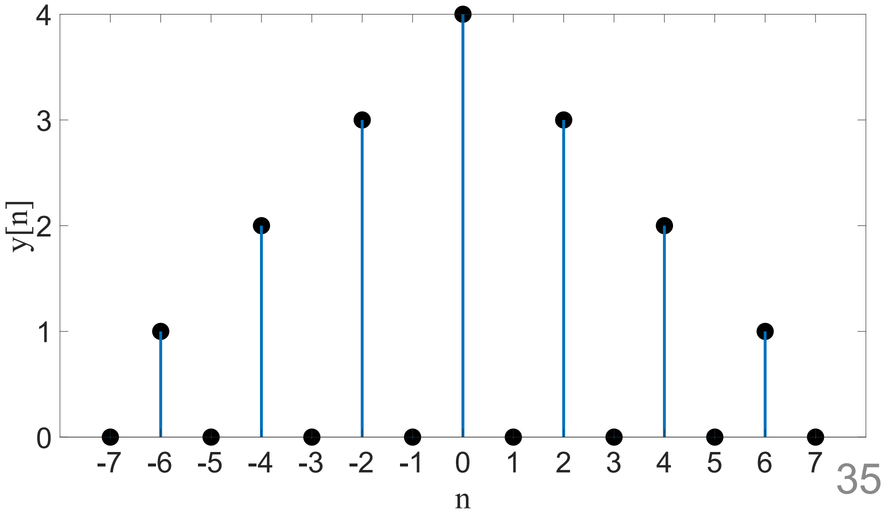
Time Scaling (Example of a DT Signal)

- Given a DT signal $x[n]$, sketch the signal $y[n]$ such that $y[n] = x[n/2]$

n	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
x	0	0	0	0	1	2	3	4	3	2	1	0	0	0	0
y	0	1	0	2	0	3	0	4	0	3	0	2	0	1	0

- $y[0] = x[0/2] = x[0] = 4$
- $y[1] = x\left[\frac{1}{2}\right] = 0$
- $y[2] = x[2/2] = x[1] = 3$
- $y[3] = x\left[\frac{3}{2}\right] = 0$
- $y[4] = x\left[\frac{4}{2}\right] = x[2] = 2$

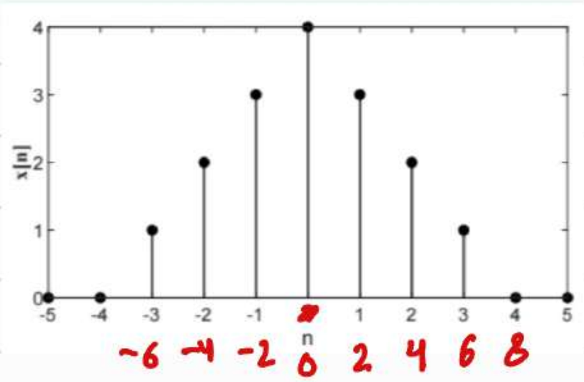




- Given a DT signal $x[n]$, sketch the signal $y[n]$ such that $y[n] = x[n/2]$

افقي ، توسيع بمقدار 2

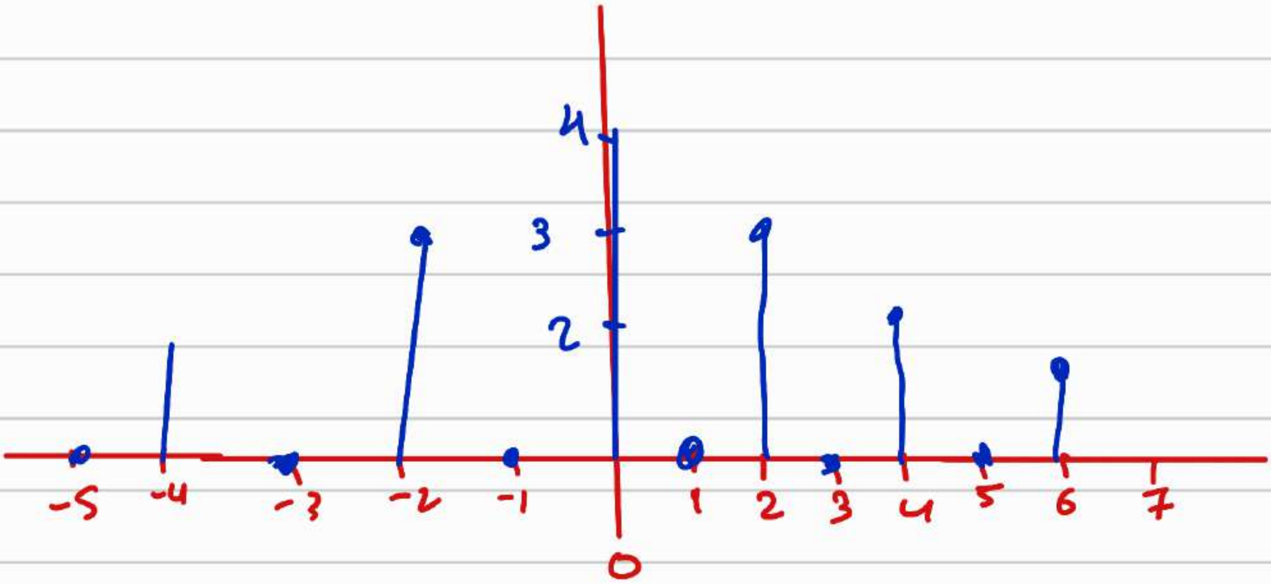
n	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
x	0	0	0	0	1	2	3	4	3	2	1	0	0	0	0
y								4	0	3					



$y[0] = x[0]$

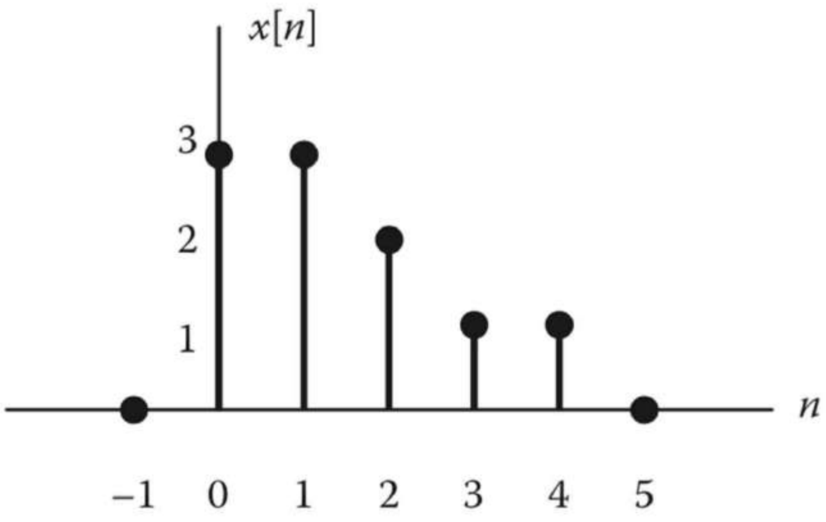
$y[1] = x[\frac{1}{2}]$

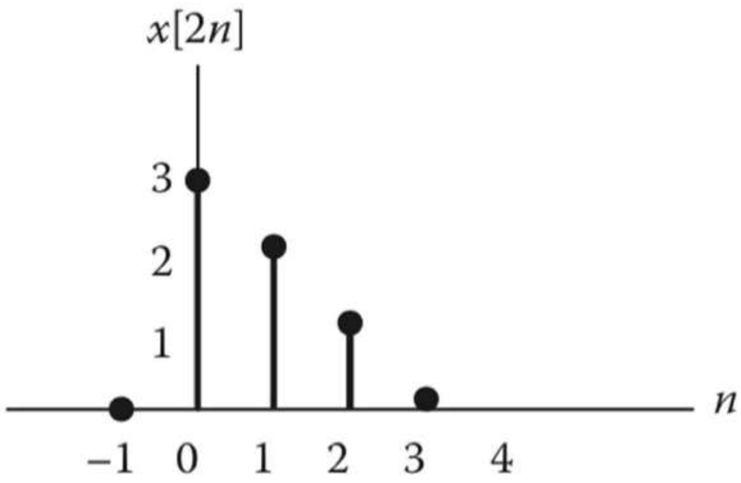
$y[2] = x[1]$

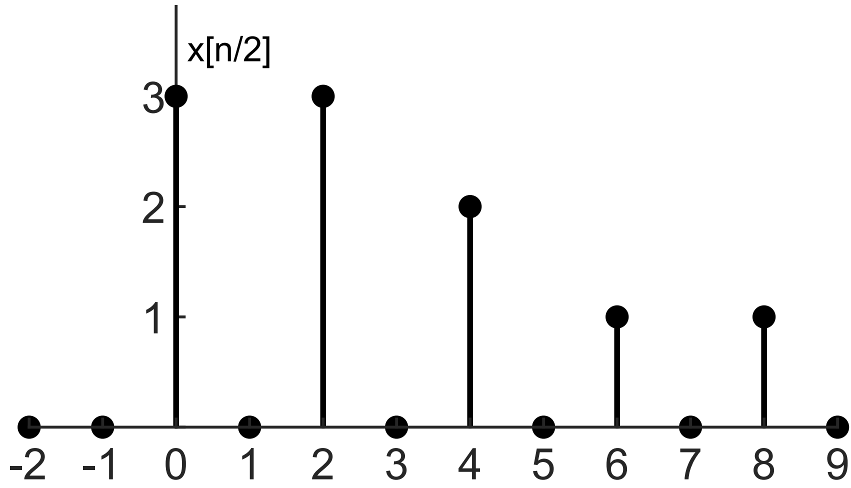


Practice Problems

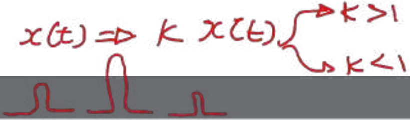
7. For the signal shown, sketch $x[2n]$ and $x[n/2]$







Amplitude Scaling of Signals



Definition

For **continuous-time (CT)** signals, amplitude (vertical) scaling maps the input $x(t)$ to

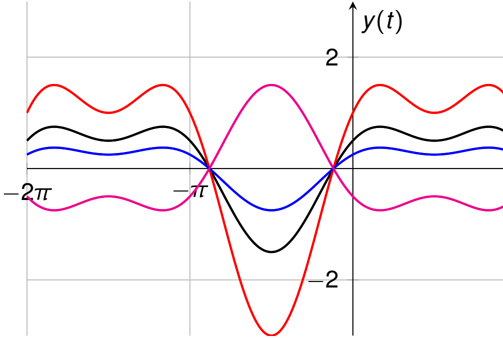
$$\boxed{y(t) = k\,x(t)}, \qquad k \in \mathbb{C} \ \text{(often } k \in \mathbb{R}\text{)}.$$

For **discrete-time (DT)** signals, amplitude scaling maps the input $x[n]$ to

$$\boxed{y[n] = k\,x[n]}, \qquad k \in \mathbb{C} \ \text{(often } k \in \mathbb{R}\text{)}.$$

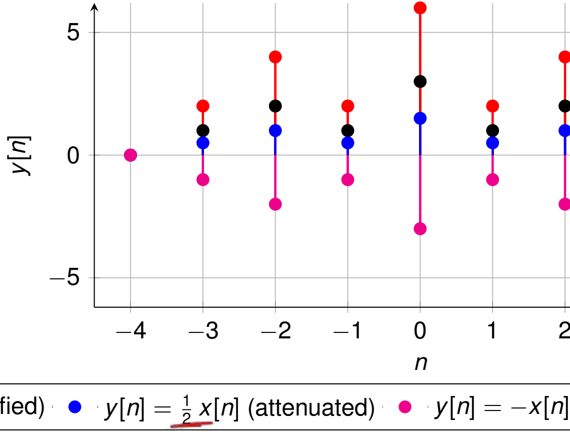
- $|k| > 1$: **amplification** (peaks increase by $|k|$).
- $0 < |k| < 1$: **attenuation** (peaks shrink by $|k|$).
- $k = 0$: output is the **zero signal**.
- $k < 0$ (real): **sign inversion** (a 180° phase flip for sinusoids). If k is complex, magnitude scales by $|k|$ and the phase is shifted by $\angle k$.

Amplitude Scaling (CT Example)



$x(t)$ — $y(t) = 2\,x(t)$ (amplified) — $y(t) = \frac{1}{2}\,x(t)$ (attenuated) — $y(t) = -x(t)$

Amplitude Scaling (DT Example)



Multiple Signal Transformations

Given $x(t)$, we want to obtain

$$\boxed{y(t) = A\,x(\frac{-t}{a} - \frac{t_0}{a})}, \qquad A \in \mathbb{R},\ a \neq 0,\ t_0 \in \mathbb{R}.$$

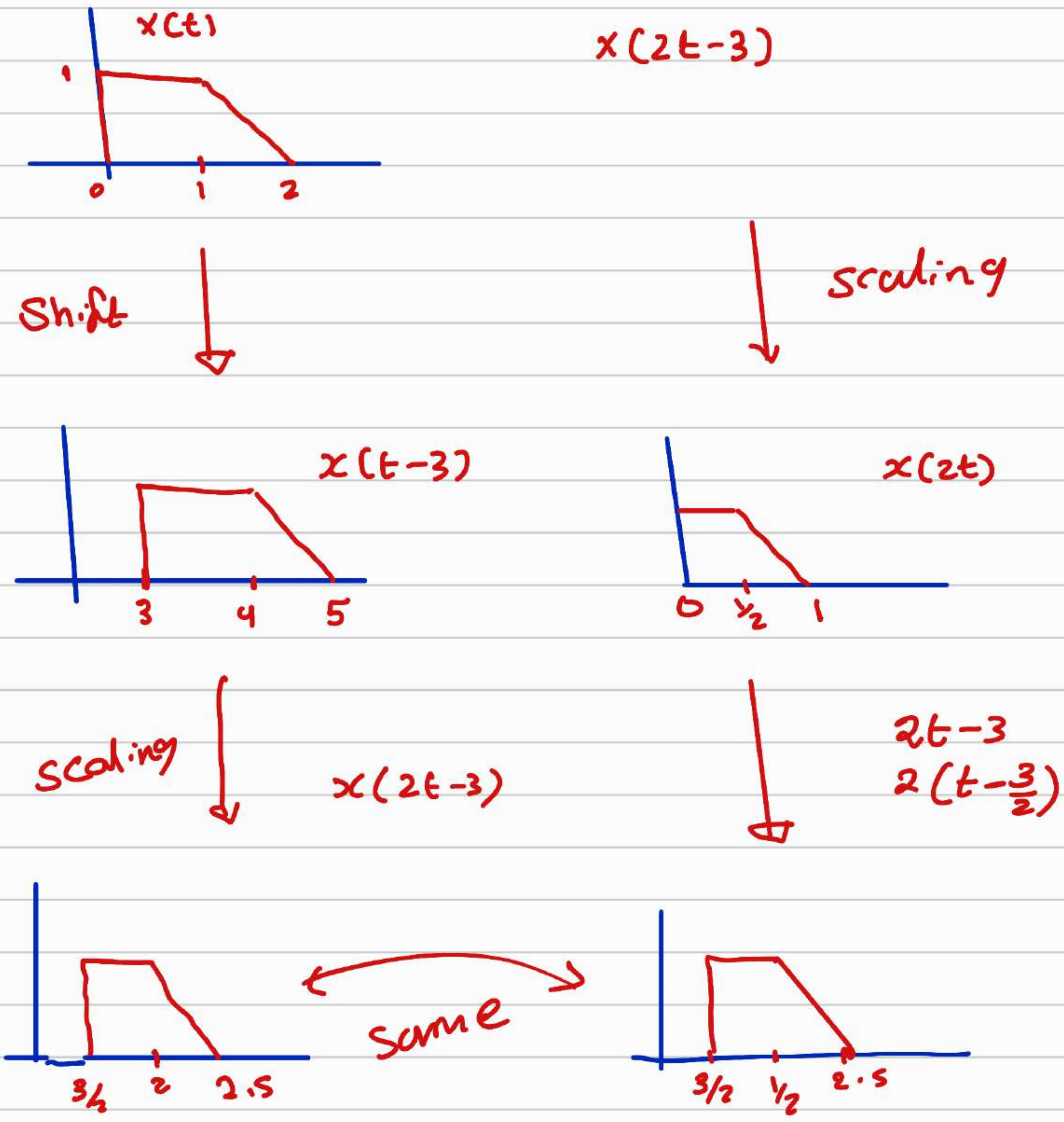
Why this form?

- It separates the three time operations cleanly:
  1. $-t$ (reversal),
  2. $-t - t_0$ (reversal + shift), and
  3. $\frac{1}{a}$ (scaling).
- Amplitude scaling (A) is independent of time operations.

reversal

$$x(3t-1) \xrightarrow{TR} x(-3t-1)$$

$$x(3t) \xrightarrow{shift} x(3(t-1))$$



Step 1: Normalize the time argument (algebra only)

Rewrite as *reversal* $\rightarrow$ *shift* $\rightarrow$ *scaling*

$$y(t) = A\,x\Big(\frac{-t-t_0}{a}\Big) = A\,x\Big(\underbrace{-t}_{\text{reversal}}\ \underbrace{-\,t_0}_{\text{shift}}\Big)\Bigg|_{t\mapsto t/a} = A\,x\Big(\frac{-t}{a} - \frac{t_0}{a}\Big).$$

Tip: First factor so every term that touches t is divided by a. That makes the three time-operations explicit and separable.

Steps 2–6: Construct $y(t)$ from $x(t)$

Let $x(t)$ be the given signal. Build $y(t)$ via the following steps:

- **(2)** Plot the **original** $x(t)$.
- **(3)** Plot $\mathbf{A\,x(t)}$ → *amplitude scaling*.
- **(4)** Plot $\mathbf{A\,x(-t)}$ → time reversal.
- **(5)** Plot $\mathbf{A\,x(-t - t_0/a)}$ → time shifting by t_0/a.
- **(6)** Plot $\mathbf{A\,x(-t/a - t_0/a)}$ → time scaling by factor $1/a$.

Important: These last three must keep their *relative order*: reversal ⇒ shift ⇒ scaling.

Keep the order

Reversal ⇒ **Shift** ⇒ **Scaling** (amplitude factor A can be applied anytime).

Example:

If $A = 2$, $t_0 = 3$, $a = 2$: $y(t) = 2\,x\left(\frac{-t-3}{2}\right)$.

(3) $2x(t)$ → (4) $2x(-t)$ → (5) $2x(-t - 1.5)$ → (6) $2x\big(-t/2 - 1.5\big)$.

Priority (must-remember order)

Time-operation priority

1) Time Reversal $\Rightarrow$ 2) Time Shifting $\Rightarrow$ 3) Time Scaling

- Perform amplitude scaling A wherever convenient.
- Following this order guarantees a correct waveform for expressions like $x\left(\frac{-t}{a} - \frac{t_0}{a}\right)$.

Why that order?

- **Reversal first:** Once the axis is flipped, left/right for later shifts is interpreted correctly.
- **Then shifting:** Apply the horizontal move by t_0/a on the already-reversed timeline.
- **Then scaling:** Stretch/compress all time marks by $1/a$.
- **Amplitude scaling:** A can be applied *anytime*–it does not affect time locations.

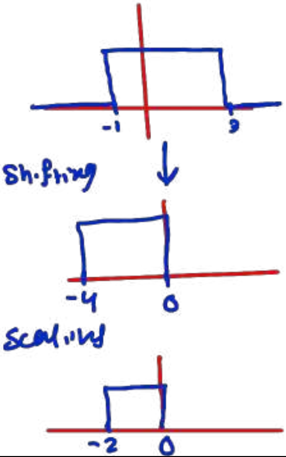
Example: Multiple Signal Operations

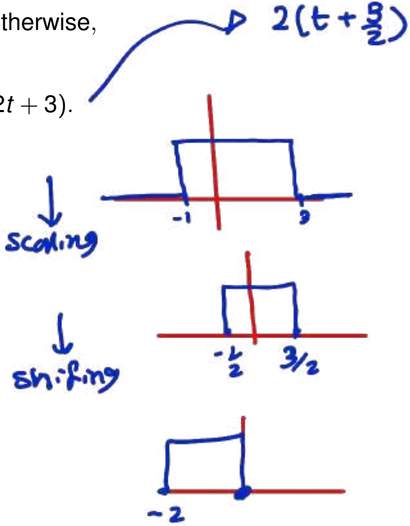
Given the input $x(t)$ (unit-amplitude pulse)

$$x(t) = \begin{cases} 1, & -1 \leq t \leq 3, \\ 0, & \text{otherwise}, \end{cases}$$

Find and Sketch

$$y(t) = x(2t + 3).$$





Method 1: Time Scaling $\Rightarrow$ Time Shifting (not recommended)

Step 1 (time scaling): $x(2t)$. Support of $x(t)$ was $[-1, 3]$. Replace $t \mapsto 2t \Rightarrow t \mapsto t/2$:

$$x(2t) = 1 \text{ on } \left[-\tfrac{1}{2}, \tfrac{3}{2}\right], \quad 0 \text{ elsewhere.}$$

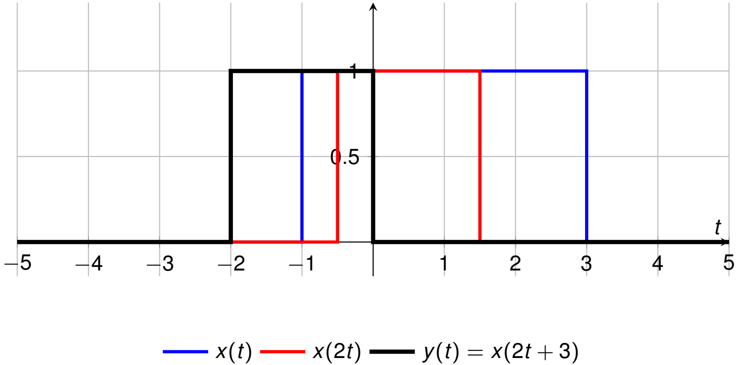
Step 2 (time shifting): $x(2t + 3) = x\big(2(t + 1.5)\big)$.

- The shift must be *against* t: left by 1.5.
- So $[-\frac{1}{2}, \frac{3}{2}] \mapsto [-2, 0]$.

$$\boxed{y(t) = 1 \text{ on } [-2, 0], \quad 0 \text{ elsewhere.}}$$

Method 1: Time Scaling $\Rightarrow$ Time Shifting

- Start with $x(t)$ on $[-1, 3]$.
- Scale in time: $x(2t) \quad (t \mapsto t/2) \Rightarrow$ support $[-\frac{1}{2}, \frac{3}{2}]$.
- Shift left by 1.5: $x(2t+3) = x\big(2(t+1.5)\big) \Rightarrow$ support $[-2, 0]$.



Method 2: Time Shifting $\Rightarrow$ Time Scaling

Step 1 (time shifting): $x(t+3)$ (left by 3).

$$x(t+3) = 1 \text{ on } [-4, 0].$$

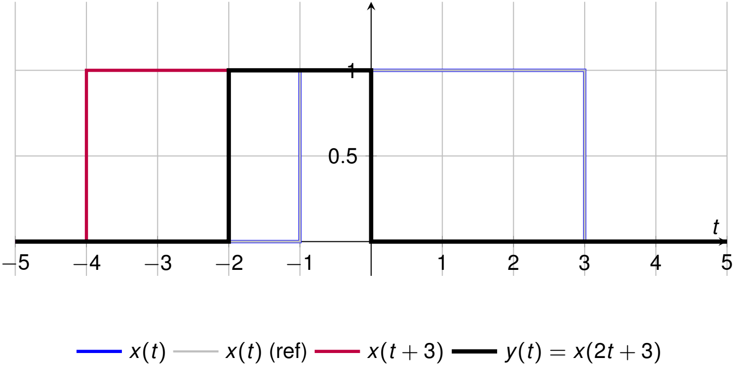
Step 2 (time scaling): now apply $t \mapsto 2t$ to obtain $x(2t+3)$.

$$[-4, 0] \xrightarrow{t \mapsto 2t} [-2, 0].$$

$$\boxed{y(t) = 1 \text{ on } [-2, 0], \;\; 0 \text{ elsewhere.}}$$

Method 2: Time Shifting $\Rightarrow$ Time Scaling

- Shift left by 3: $x(t+3) \Rightarrow$ support $[-4, 0]$.
- Scale in time: $x(2t+3) \quad (t \mapsto t/2) \Rightarrow$ support $[-2, 0]$.
- Same final result as Method 1.



Method 3 (Shortcut): Axis Substitution

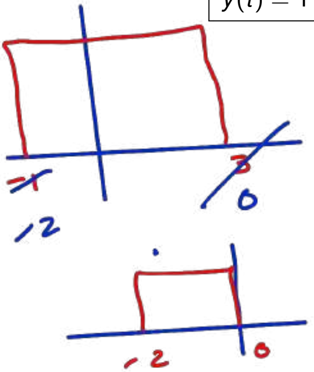
Let $s = 2t + 3$. Then $y(t) = x(s)$ has *the same shape as* x on the s-axis:

$$x(s) = 1 \text{ on } [-1, 3].$$

Convert the axis back to t: $t = \dfrac{s-3}{2}$.

$$s = -1 \Rightarrow t = -2, \qquad s = 3 \Rightarrow t = 0.$$

$$\boxed{y(t) = 1 \text{ on } [-2, 0], \;\; 0 \text{ elsewhere.}}$$



$x(2t+3)$

$2t+3=3$

$2t=0 \quad t=0$

$2t+3=-1$

$2t=-4 \quad t=-2$

Algebraic Check

$$y(t) = x(2t+3), \quad x(\cdot) = 1 \text{ on } [-1, 3].$$

$$-1 \le 2t+3 \le 3 \iff -4 \le 2t \le 0 \iff \boxed{-2 \le t \le 0}.$$

So $y(t) = 1$ on $[-2, 0]$ and 0 elsewhere (amplitude unchanged).

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Signals & Systems

1. Signals and Systems