CHAPTER 12 Derivatives for Functions of Two or More Variables

$f(x) = y = x^2 + 2$ one variable

$f(x,y) = z = x^2 + y - 4$ two variables

$f(x,y,z) = w = x^2 + y^2 + 3z^2$ 3 variables

* Find the domain of the one variable funcn--

$y = x^2 + 1$ $\quad D: (-\infty, \infty)$ or $\mathbb{R}$

$y = \sqrt{x+3}$ $\quad x + 3 \geq 0 \quad x \geq -3$

$D: \{x : x \geq -3\}$

$y = \dfrac{1}{x-2}$ $\quad x - 2 = 0 \quad x = 2$

$D: \{x : x \neq 2\}$

$\mathbb{R} - \{2\}$

الدالة ذي متغيرين سه لها مجال لـ x ومجال لـ Y
ويكونوا المجال شكلهم مساحة

Find the domain of the following functions and sketch

a) $z = x^2 + y^2 - 3$ $\qquad D = \{(x,y) : x,y \in \mathbb{R}\}$

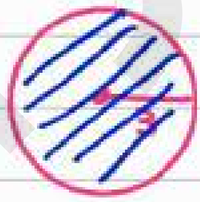
$\underset{\mathbb{R}}{x^2}$, $\underset{\mathbb{R}}{y^2}$ $\qquad \mathbb{R}^2$

b) $z = \sqrt{9 - x^2 - y^2}$

$9 - x^2 - y^2 \geq 0 \qquad 9 \geq x^2 + y^2$

$D = \quad x^2 + y^2 \leq 9$

$x^2 + y^2 = 9$



مجال صند الداله هو جميع النقاط داخل صند الدائرة

Examples: Find and sketch the domain of each function:

1. $f(x,y) = x^3 + 2xy + y^2$
2. $f(x,y) = \frac{\sqrt{x^2+y^2-9}}{x}$
3. $f(x,y) = \ln xy$

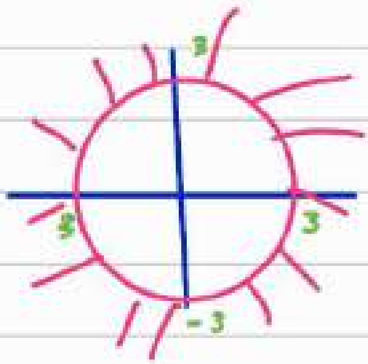
1) $f(x,y) = x^3 + 2xy + y^2$

$$D = \{(x,y) : (x,y) \in R\}$$

$$D = R^2$$

2) $f(x,y) = \frac{\sqrt{x^2+y^2-9}}{x}$

$x^2+y^2-9 \geq 0$ $\qquad$ $x^2+y^2 \overset{=}{\cancel{\geq}} 9$

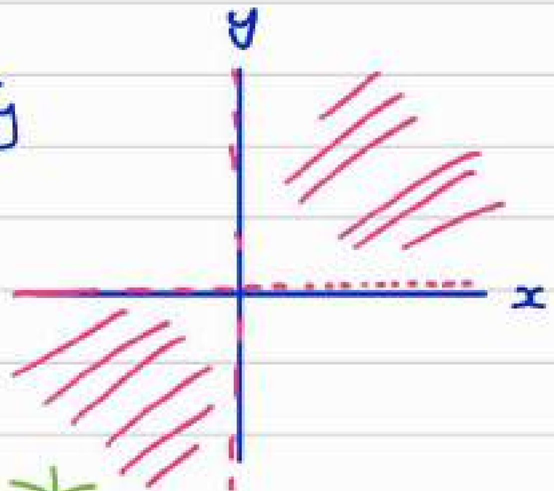


$x \neq 0$

$$D = \{x \neq 0,\ x^2+y^2 \geq 9\}$$

3) $f(x,y) = \ln xy$

$$D = \{(x,y) : xy > 0\}$$



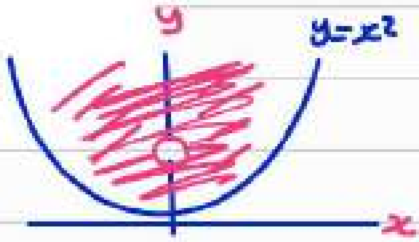
$x^2+y^2 = r^2$ circle

$x^2+y^2+z^2 = r^2$ sphere

$y = x^2$ parabola $\qquad$ $x^2 = y$

EXAMPLE 1 In the xy-plane, sketch the natural domain for

$$f(x, y) = \frac{\sqrt{y - x^2}}{x^2 + (y-1)^2}$$



$y - x^2 \geq 0 \qquad y \geq x^2$

$x^2 + (y-1)^2 = 0 \qquad (0,1)$ المقام يساوي صفر عند

$D = \{(x,y) : y \geq x^2, (x,y) \neq (0,1)\}$

we must exclude $\{(x,y) : y < x^2, (x,y) = (0,1)\}$

EXAMPLE 6 Find the domain of each function and describe the level surfaces for f.

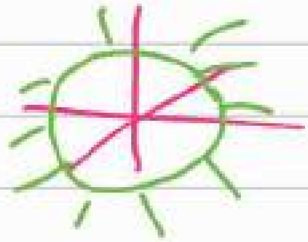
(a) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 1}$

(b) $g(w, x, y, z) = \dfrac{1}{\sqrt{w^2 + x^2 + y^2 + z^2 - 1}}$

a) $x^2 + y^2 + z^2 - 1 \geq 0$

$x^2 + y^2 + z^2 \geq 1$

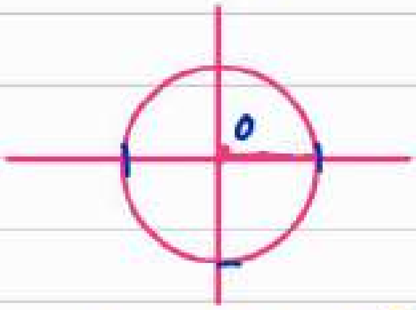
كره نصف قطرها 1



b) $w^2 + x^2 + y^2 + z^2 - 1 > 0$

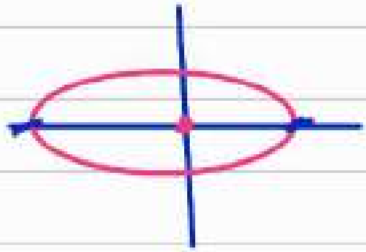
$w^2 + x^2 + y^2 + z^2 > 0$

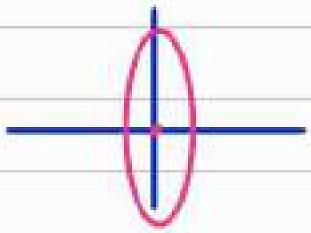
Circle الدائرة



$x^2 + y^2 = r^2$

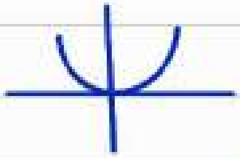
Ellipse



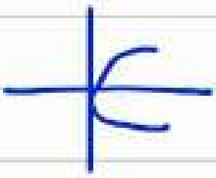


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parabola

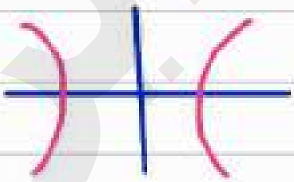


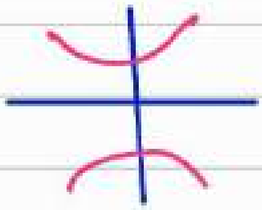
$y = kx^2$



$x = ky^2$

Hyperbola





$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Equation of a Sphere

Center at Origin:

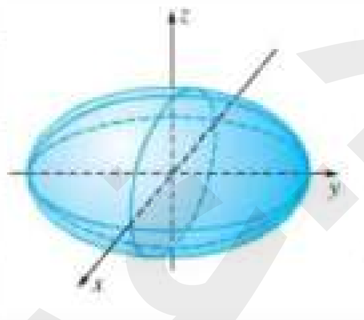
$$\boxed{x^2 + y^2 + z^2 = r^2}$$

كرة

$x^2 + y^2 + z^2 = 2^2$

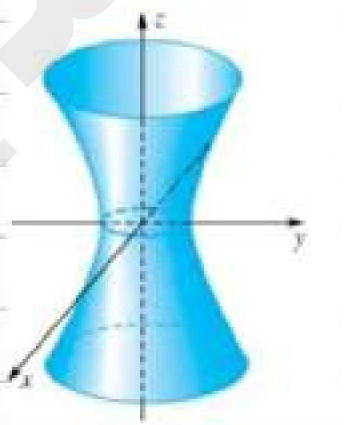
QUADRIC SURFACES

ELLIPSOID: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



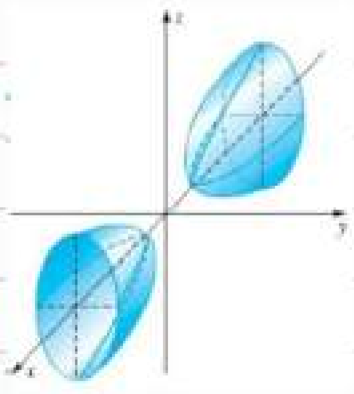
$3x^2 + 5y^2 + 7z^2 = 1$

HYPERBOLOID OF ONE SHEET: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



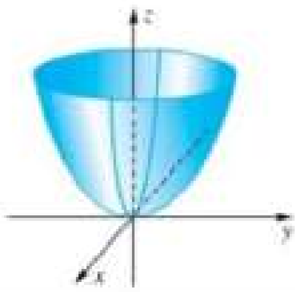
$10x^2 + 9y^2 - 6z^2 = 1$

HYPERBOLOID OF TWO SHEETS: $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



$-10x^2 - 9y^2 + 6z^2 = 1$

ELLIPTIC PARABOLOID: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



$x = 10y^2 + 20z^2$

EXAMPLE 2 Sketch the graph of $f(x, y) = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$.

$Z \geqslant 0$ يجب ان تكون قيم Z موجبة

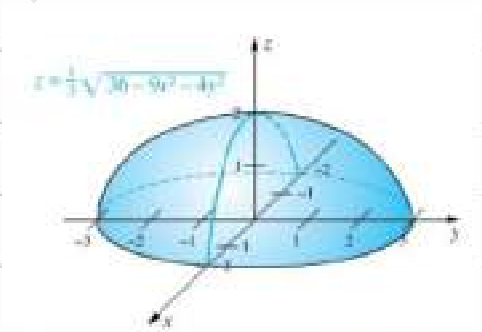
$$Z = \frac{1}{3}\sqrt{36 - 9x^2 - 4y^2}$$

$$(3Z)^2 = \left(\sqrt{36 - 9x^2 - 4y^2}\right)^2$$

$$9Z^2 = 36 - 9x^2 - 4y^2$$

$$9x^2 + 4y^2 + 9z^2 = 36$$

Ellipsoid



$$\frac{9}{36}x^2 + \frac{4}{36}y^2 + \frac{9}{36}z^2 = 1$$

$$\boxed{\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1}$$

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$$

Problem Set 12.1

1. Let $f(x, y) = x^2y + \sqrt{y}$. Find each value.

(a) $f(2, 1)$ (b) $f(3, 0)$
(c) $f(1, 4)$ (d) $f(a, a^4)$
(e) $f(1/x, x^4)$ (f) $f(2, -4)$

What is the natural domain for this function?

2. Let $f(x, y) = y/x + xy$. Find each value.

(a) $f(1, 2)$ (b) $f(\frac{1}{4}, 4)$
(c) $f(4, \frac{1}{4})$ (d) $f(a, a)$
(e) $f(1/x, x^2)$ (f) $f(0, 0)$

What is the natural domain for this function?

3. Let $g(x, y, z) = x^2 \sin yz$. Find each value.

(a) $g(1, \pi, 2)$ (b) $g(2, 1, \pi/6)$

① $f(x,y) = x^2y + \sqrt{y}$

a) $f(2,1) = (2)^2(1) + \sqrt{1} = 5$

b) $f(3,0) = 3^2(0) + \sqrt{0} = 0$

Domian $y \geq 0$

$D = \{(x, y) : x \in \mathbb{R} \text{ and } y \geq 0$

set of all numbers such that y is nonnegtive

c) $f(1,4) = x^2y + \sqrt{y}$

$= 1(4) + \sqrt{4} = 6$

d) $f(a, a^4) = a^2 a^4 + \sqrt{a^4}$.

$= a^6 + a^2$

e) $f(\frac{1}{x}, x^4) = \frac{1}{x^2} x^4 + \sqrt{x^4}$

$= x^2 + x^2 = 2x^2$

f) $f(2,-4) = (2)^2(-4) + \sqrt{\cancel{-4}}$

undefined

2) $f(x,y) = \frac{y}{x} + xy$

a) $f(1,2) = \frac{2}{1} + 1(2) = 4$

b) $f(\frac{1}{4}, 4) = \frac{4}{1/4} + 4 \times \frac{1}{4} = 16 + 1 = 17$

c) $f(4, \frac{1}{4}) = \frac{\frac{1}{4}}{4} + \cancel{4} \cdot \frac{1}{\cancel{4}} = \frac{1}{16} + 1 = \frac{17}{16}$

d) $f(a,a) = \frac{a}{a} + a \cdot a = 1 + a^2$

the set of all (x,y) such that $x = 0$

e) $f(\frac{1}{x}, x^2) = \frac{x^2}{1/x} + \frac{1}{x} x^2 = x^3 + x$

f) $f(0,0) =$ undefined

3) $g(x,y,z) = x^2 \sin xy$

$g(1,\pi,2) = \sin 2\pi = 0$

$g(2,1,\frac{\pi}{6}) = 4 \sin \frac{\pi}{6} = 2$

5. Find $F(f(t), g(t))$ if $F(x, y) = x^2y$ and $f(t) = t\cos t$, $g(t) = \sec^2 t$.

$F(x,y) = x^2 y$

$F(f(t), g(t)) = f^2(t) \cdot g(t)$

$= t^2 \cos^2 t \sec^2 t$

$= t^2 \cancel{\cos^2 t} \, \frac{1}{\cancel{\cos^2 t}}$

$= t^2$

6. Find $F(f(t), g(t))$ if $F(x, y) = e^x + y^2$ and $f(t) = \ln t^2$, $g(t) = e^{t/2}$.

$$F(x,y) = e^x + y^2$$

$t \neq 0$

$$F(f(t), g(t)) = e^{f(t)} - (g(t))^2$$

$$F(f(t), g(t)) = e^{\ln t^2} + (e^{t/2})^2$$

$$t^2 + e^t$$

12.2
Partial Derivatives

سابقاً

$f(x) = y$ ——— مشتقة

$$f'(x) \quad f'' \quad f'''$$
$$y' \quad y''$$
$$\frac{dy}{dx} \quad \frac{d^2y}{dx^2} \quad \frac{d^3}{dx^3}$$

$$f(x) = y = 3x^2+2 \qquad f' = y' = \frac{dy}{dx} = 6x$$

$f(x,y) = z$

$f_x(x,y)$, $\frac{\partial z}{\partial x}$ نشتق الداله بالنسبة لـ x ونعتبر y ثابت

$f_y(x,y)$, $\frac{\partial z}{\partial y}$ نشتق بالنسبة لـ y ونعتبر x ثابت

مثال

* $f(x,y) = 3x^2 + 2y$

$f_x = 6x \qquad f_y = 2$

* $f(x,y) = x^3 + y^2 + 2xy$

$f_x = 3x^2 + 2y \qquad f_y = 2y + 2x$

$f(x,y)$	f_x	f_y
$f(x,z) = e^{xy}$	$f_x = ye^{xy}$	$f_y = xe^{xy}$
$f(x,y) = x^2y$	$f_x = 2yx$	$f_y = x^2$
$f(x,y) = x^2 \sin y$	$f_x = 2x \sin y$	$f_y = x^2 \cos y$
$f(x,y) = \sqrt{x^2 - 4y}$ $= (x^2 - 4y)^{1/2}$	$f_x = \frac{1}{2}(x^2 - 4y)^{-1/2} \cdot (2x)$ $f_y = \frac{1}{2}(x^2 - 4y)^{-1/2} \cdot (-4)$	
$f(x,y) = x \cos xy$	$f_x = -xy \sin xy + (1) \cos xy$ $f_y = -x^2 \sin xy$	
$f(x,y) = xy^{-2}$	$f_x = y^{-2}$ $f_x = \frac{1}{y^2}$	$f_y = x(-2y^{-3})$ $f_y = \frac{-2x}{y^3}$
$f(x,y) = x^3 + y^2 + xy$ $f_x(1,0)$	$f_x = 3x^2 + y$ $f_x(1,0) = 3 + 0 = 3$	

Suppose that f is a function of two variables x and y. If y is held constant, say $y = y_0$, then $f(x, y_0)$ is a function of the single variable x. Its derivative at $x = x_0$ is called the **partial derivative of f with respect to x** at (x_0, y_0) and is denoted by $f_x(x_0, y_0)$. Thus,

$$\boxed{f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}}$$

تعريف المشتقة الجزئية

Similarly, the partial derivative of f with respect to y at (x_0, y_0) is denoted by $f_y(x_0, y_0)$ and is given by

$$\boxed{f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}}$$

EXAMPLE 1 Find $f_x(1, 2)$ and $f_y(1, 2)$ if $f(x, y) = x^2y + 3y^3$.

$$f(x,y) = x^2y + 3y^3$$

$$f_x = 2xy$$

$$f_x(1,2) = 2(1)(2) = 4$$

$$f_y = x^2 + 9y^2$$

$$f_y(1,2) = 1^2 + 9(2)^2 = 37$$

EXAMPLE 2 If $z = x^2 \sin(xy^2)$, find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\frac{\partial z}{\partial x} = x^2 . \cos(xy^2) . y^2 + 2x \sin(xy^2)$$

$$\frac{\partial z}{\partial x} = x^2y^2 \cos(xy^2) + 2x \sin(xy^2)$$

$$\frac{\partial z}{\partial y} = x^2 \cos(xy^2) . 2xy$$

$$= 2x^3y \cos(xy^2)$$

Higher Partial Derivatives

f_x $\quad \frac{\partial z}{\partial x}$ $\quad$ مشتقة أولى $\quad$ f_y $\quad \frac{\partial z}{\partial y}$

f_{xx} نشتق اول مره بالنسبه لـ x و مره الثانيه بالنسبه لـ x

$$f_{xx} = \frac{d^2f}{dx\,dx} = \frac{d^2f}{dx^2}$$

f_{yy} نشتق اول مره بالنسبه لـ y و مره الثانيه بالنسبه لـ y

$$f_{yy} = \frac{d^2f}{dy\,dy} = \frac{d^2f}{dy^2}$$

f_{xy} نشتق اول مره بالنسبه لـ x ثم مره الثانيه بالنسبه لـ y

$$f_{xy} = \frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dy\,dx}$$

f_{yx} نشتق اول مره بالنسبه لـ y ثم بالنسبه لـ x

$$f_{xy} = \frac{d}{dx}\left(\frac{df}{dy}\right) = \frac{d^2 f}{dx\,dy}$$

مثال :

$$f(x,y) = 2x^3 + 3xy^2 + y^4$$

$f_x = 6x^2 + 3y^2$ $\qquad$ $f_y = 6xy + 4y^3$

$f_{xx} = 12x$

$f_{yy} = 6x + 12y^2$

$f_{xy} = 6y$

$f_{yx} = 6y$

f_{xx} $\quad$ f_{yy} $\quad$ f_{xy} $\quad$ f_{yx}

EXAMPLE 5 Find the four second partial derivatives of

$$f(x, y) = xe^y - \sin(x/y) + x^3y^2$$

$$f(x,y) = xe^y - \sin\left(\tfrac{1}{y}x\right) + x^3y^2$$

$$f_x = e^y - \frac{1}{y}\cos\left(\tfrac{1}{y}x\right) + 3x^2y^2$$

$$f_y = xe^y + \frac{x}{y^2}\cos\left(\tfrac{1}{y}x\right) + 2x^3y$$

$$\boxed{f_{xx} = +\frac{1}{y^2}\sin\left(\tfrac{1}{y}x\right) + 6xy^2}$$

$$f_{yy} = xe^y - \frac{x}{y^2}\sin\left(\tfrac{1}{y}x\right)\left(-\frac{x}{y^2}\right) - \frac{2x}{y^3}\cos\left(\tfrac{1}{y}x\right) + 2x^3$$

$$\boxed{f_{yy} = xe^y + \frac{x^2}{y^4}\sin\left(\frac{x}{y}\right) - \frac{2x}{y^3}\cos\frac{x}{y} + 2x^3}$$

$$f_{xy} = e^y + \frac{1}{y}\sin\left(\tfrac{1}{y}x\right)\left(\frac{-x}{y^2}\right) + \frac{1}{y^2}\cos\frac{x}{y} + 6x^2y$$

$$\boxed{f_{xy} = e^y - \frac{x}{y^3}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\cos\frac{x}{y} + 6x^2y}$$

$$f_{yx} = e^y - \frac{x}{y^2}\sin\left(\tfrac{1}{y}x\right)\frac{1}{y} + \frac{1}{y^2}\cos\left(\tfrac{1}{y}x\right) + 6x^2y$$

$$\boxed{f_{yx} = e^y - \frac{x}{y^3}\sin\left(\frac{x}{y}\right) + \frac{1}{y^2}\cos\left(\frac{x}{y}\right) + 6x^2y}$$

داله فيها 3 متغيرات

EXAMPLE 6 If $f(x, y, z) = xy + 2yz + 3zx$, find f_x, f_y, and f_z.

عند وجود داله عدديه المتغيرات نشتق المطلوب ونعتبر كل الباقي ثوابت

$$f(x,y,z) = xy + 2yz + 3zx$$

$$f_x = y + 3z \qquad f_y = x + 2z$$

$$f_z = 2y + 3x$$

EXAMPLE 7 If $T(w, x, y, z) = ze^{w^2+x^2+y^2}$, find all first partial derivatives and $\frac{\partial^2 T}{\partial w\, \partial x}$, $\frac{\partial^2 T}{\partial x\, \partial w}$, and $\frac{\partial^2 T}{\partial z^2}$.

$$\frac{\partial T}{\partial w} = ze^{w^2+x^2+y^2}(2w) = 2zwe^{w^2+x^2+y^2}$$

$$\frac{\partial T}{\partial x} = ze^{w^2+x^2+y^2}(2x) = 2zxe^{w^2+x^2+y^2}$$

$$\frac{\partial T}{\partial y} = ze^{w^2+x^2+y^2}(2y) = 2zye^{w^2+x^2+y^2}$$

$$\frac{\partial T}{\partial z} = e^{w^2+x^2+y^2}$$

اول مره نشتق لـ x ثم نشتق بالنسبة لـ w

$$\frac{\partial^2 T}{\partial w \partial x} = 2xz\,e^{w^2+x^2+y^2}(2w)$$
$$= 4xzw\,e^{w^2+x^2+y^2}$$

$$\frac{\partial^2 T}{\partial x \partial w} = 2zw\,e^{w^2+x^2+y^2}(2x)$$
$$= 4zwx\,e^{w^2+x^2+y^2}$$

$$\frac{\partial^2 T}{\partial z^2} = 0$$

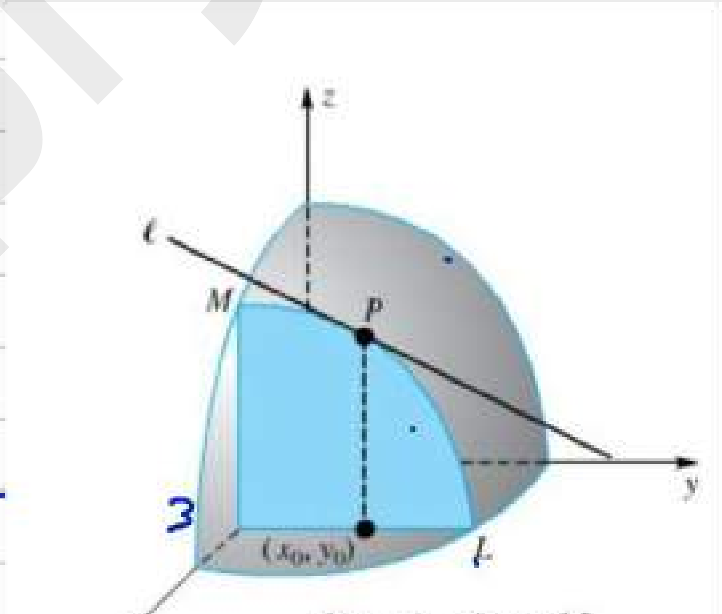
Geometric and Physical Interpretations

$z = f(x, y)$

f_y ميل الخط

في حاله تثبيت قيمه x_0

$\frac{\partial z}{\partial y}$ المتغير الرأسي



$f_y(x_0, y_0)$ = slope of ℓ

في حال تثبيت y_0

f_x = ميل الخط

$\frac{\partial z}{\partial x}$ التغير الرأسي

EXAMPLE 3 The surface $z = f(x, y) = \sqrt{9 - 2x^2 - y^2}$ and the plane $y = 1$ intersect in a curve as in Figure 1. Find parametric equations for the tangent line at $(\sqrt{2}, 1, 2)$.

$\underset{x}{} \ \underset{y}{} \ \underset{z}{}$

ميل العماس عند النقطة

تم تثبيت y

$(\sqrt{2}, 1, 2)$ ميل المماس عند النقطة

يساوي f_x

$$z = (9 - 2x^2 - y^2)^{1/2}$$

$$f_x = \frac{1}{2}(9 - 2x^2 - y^2)^{-1/2}(-4x)$$

$$f_x = \frac{-4x}{2\sqrt{9 - 2x^2 - y^2}}$$

حساب الميل عند النقطة المعطاة

$(\sqrt{2}, 1, 2)$

$$f(x)(\sqrt{2}, 1) = \frac{-4\sqrt{2}}{2\sqrt{9 - 4 - 1}} = -\frac{4\sqrt{2}}{4}$$

$$f_x = -\sqrt{2}$$

$(\sqrt{2}, 1, 2)$

الاتجاه

$$\left(1, 0, \frac{2}{-\sqrt{2}}\right)$$

$$(1, 0, -\sqrt{2})$$

$$y = 1$$

$$x = \sqrt{2} + t$$

$$z = 2 + -\sqrt{2}\, t$$

$$x = x_0 + d_x(t)$$
$$z = z_0 + d_z(t)$$
$$y = y_0 + d_y(t)$$

Problem Set 12.2

In Problems 1–16, find all first partial derivatives of each function.

1. $f(x, y) = (2x - y)^4$

$$f_x = 4(2x-y)^3(2) = 8(2x-y)^3$$

$$f_y = 4(2x-y)^3(-1) = -4(2x-y)^3$$

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - g' f}{g^2}$$

3. $f(x, y) = \dfrac{x^2 - y^2}{xy}$

$$f_x = \frac{(xy)\cdot(2x) - (y)(x^2-y^2)}{(xy)^2}$$

$$= \frac{2x^2y - x^2y + y^3}{x^2y^2} = \frac{x^2y + y^3}{x^2y} = \frac{x^2+y^2}{x^2y}$$

$$f_y = \frac{(xy)(-2y) - x(x^2-y^2)}{x^2y^2} = \frac{x^2+y^2}{xy^2}$$

5. $f(x, y) = e^y \sin x$

$f_x = e^y \cos x$ $\qquad$ $f_y = e^y \sin x$

7. $f(x, y) = \sqrt{x^2 - y^2} = (x^2 - y^2)^{1/2}$

$$f_x = \frac{1}{2}(x^2 - y^2)^{-1/2} \cdot (2x) = x(x^2 - y^2)^{-1/2}$$

$$f_y = \frac{1}{2}(x^2 - y^2)^{-1/2}(-2y) = -y(x^2 - y^2)^{-1/2}$$

9. $g(x, y) = e^{-xy}$

$$g_x = -ye^{-xy} \qquad g_y = -x\,e^{-xy}$$

13. $f(x, y) = y\cos(x^2 + y^2)$

$$f_x = -y\sin(x^2 + y^2) \cdot 2x = -2xy\sin(x^2 + y^2)$$

$$f_y = -y\sin(x^2 + y^2) \cdot (2y) + \cos(x^2 + y^2)$$

$$= -2y^2\sin(x^2 + y^2) + \cos(x^2 + y^2)$$

In Problems 17–20, verify that

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$$

17. $f(x, y) = 2x^2y^3 - x^3y^5$

$f_x = 4xy^3 - 3x^2y^5$

$f_y = 6x^2y^2 - 5x^3y^4$

$f_{xy} = 12xy^2 - 15x^2y^4$

$f_{yx} = 12xy^2 - 15x^2y^4$

$$f_{xy} = f_{yx}$$

19. $f(x, y) = 3e^{2x} \cos y$

$f_x = 3e^{2x}(2)\cos y$

$f_y = -3e^{2x}\sin y$

$f_x = 6e^{2x}\cos y$

$f_{xy} = -6e^{2x}\sin x$

$f_{yx} = -3e^{2x}(2)\sin y$

$f_{yx} = -6e^{2x}\sin y$

$$f_{xy} = f_{yx}$$

21. If $F(x, y) = \dfrac{2x - y}{xy}$, find $F_x(3, -2)$ and $F_y(3, -2)$.

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$F_x = \frac{(xy)(2) - (2x-y)y}{x^2y^2}$$

$$F_x = \frac{\cancel{2xy} - \cancel{2xy} + y^2}{x^2y^2} = \frac{\cancel{y^2}}{x^2\cancel{y^2}} = \frac{1}{x^2}$$

$$F_x(3,-2) = \frac{1}{3^2} = \frac{1}{9}$$

$$F_y = \frac{xy(-1) - (2x-y)(x)}{x^2y^2} = \frac{\cancel{-xy} - 2x^2 \cancel{+xy}}{x^2y^2}$$

$$\frac{-2x^{\cancel{2}}}{\cancel{x^2}y^2} = \frac{-2}{y^2}$$

$$F_y(3,-2) = \frac{-2}{(-2)^2} = -\frac{1}{2}$$

11. $f(x, y) = \tan^{-1}(4x - 7y)$

$$f_x = \frac{4}{1+(4x-7y)^2} \qquad f_y = \frac{-7}{1+(4x-7y)^2}$$

15. $F(x, y) = 2 \sin x \cos y$

$$f_x = 2\cos y \cos x$$

$$f_y = -2\sin x \sin y$$

In Problems 17–20, verify that

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$$

18. $f(x, y) = (x^3 + y^2)^5$

$$f_x = 5(x^3+y^2)^4(3x^2) = 15(x^3+y^2)^4 x^2$$

$$f_{xy} = 15 \cdot 4(x^3+y^2)^3 \cdot (2y)x^2 = 120(x^3+y^2)yx^2$$

$$f_{xy} = 120x^2y(x^3+y^2)$$

$$f_y = 5(x^3+y^2)^4(2y) = 10y(x^3+y^2)^4$$

$$f_{yx} = 10y \cdot 4 \cdot (x^3+y^2)^3\, 3x^2 = 120x^2y(x^3+y^2)^2$$

20. $f(x, y) = \tan^{-1} xy$

$$f_x = \frac{y}{1+x^2y^2} = y(1+x^2y^2)^{-1}$$

$$f_{xy} = y\left[-(1+x^2y^2)^{-2} \cdot 2x^2y\right] + (1+x^2y^2)^{-1}$$

$$= -2x^2y^2(1+x^2y^2)^{-2} + (1+x^2y^2)^{-1}$$

$$(1+x^2y^2)^{-2}(-2x^2y^2 + 1 + x^2y^2)$$

$$(1+x^2y^2)^{-2}(1-x^2y^2)$$

$$f_y = \frac{x}{1+x^2y^2} = x(1+x^2y^2)^{-1}$$

$$f_{yx} = x(1+x^2y^2)^{-2}(2xy^2)$$

$$+ (1+x^2y^2)^{-1}$$

$$f_{xy} = -2x^2y^2(1+x^2y^2)^{-2} + (1+x^2y^2)^{-1}$$

$$= (1+x^2y^2)^{-2}(-2x^2y^2 + 1 + x^2y^2)$$

$$= (1+x^2y^2)^{-2}(1-x^2y^2)$$

23. If $f(x, y) = \tan^{-1}(y^2/x)$, find $f_x(\sqrt{5}, -2)$ and $f_y(\sqrt{5}, -2)$.

$$f_x = \frac{1}{\left(1 + \frac{y^4}{x^2}\right)} \cdot \frac{-y^2}{x^2}$$

$$\frac{1}{x}$$

$$-\frac{1}{x^2}$$

$$= \frac{-y^2}{x^2 + y^4}$$

$$f_x(\sqrt{5}, -2) = \frac{-(2)^2}{(\sqrt{5})^2 + (-2)^4} = -\frac{4}{21}$$

$$f_y = \frac{1}{1 + \frac{y^4}{x^2}} \cdot \frac{2y}{x} = \frac{2y}{x + \frac{y^4}{x}} = \frac{2xy}{x^2 + y^4}$$

$$f(\sqrt{5}, -2) = -\frac{4\sqrt{5}}{21}$$

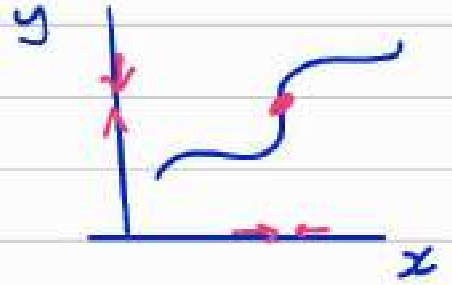
24. If $f(x, y) = e^y \cosh x$, find $f_x(-1, 1)$ and $f_y(-1, 1)$.

$$f_x = e^y \sinh x \Rightarrow f_x(-1,1) = e^1 \sinh(-1)$$

$$= e \sinh(-1)$$

$$f_y = e^y \cosh x = e \cosh(-1)$$

12.3
Limits and Continuity



$$f(x,y) = 3x + 4y^2$$

قواعد النهايات للدوال ثنائية المتغيرات

① اذا كانت الداله كثيره حدود فقط نعوض x,y
مباشره تحصل النهايه

$$\lim_{(0,1)} 3x + 4y^2 = 0 + 4 = 4$$

② اذا كانت الداله كسرية

$$f(x,y) = \frac{p}{q}$$

بعد التعويض

$\frac{\text{رقم}}{\text{صفر}}$

النهايه غير موجوده

بعد التعويض

$= \frac{0}{0}$

يجب دراسه هذه الداله بشكل
افضل لنتوقع وجود النهايه

علينا تثبيت x وحساب
نهايه x
ثم تثبيت y وحساب
نهايه y
اذا تساوت النهايتين تكون النهايه موجود

علينا تحليل الداله
للتخلص من الأصفار

EXAMPLE 1 Evaluate the following limits if they exist:

(a) $\lim_{(x,y)\to(1,2)} (x^2y + 3y)$ and (b) $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + y^2 + 1}{x^2 - y^2}$

a) $\lim_{(x,y)\to(1,2)} x^2y + 3y = 1^2(2) + 3(2) = 6$

b) $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2+1}{x^2-y^2} = \dfrac{0+0+1}{0-0} = \dfrac{1}{0}$

limit does not exist

EXAMPLE 2 Show that the function f defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

has no limit at the origin (Figure 3).

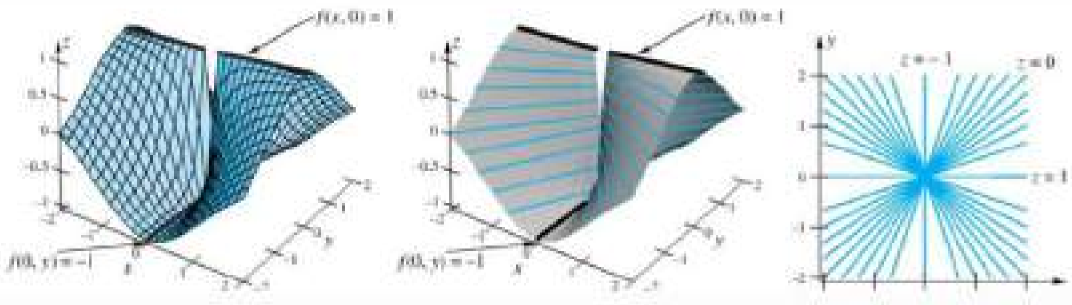
$\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \dfrac{0-0}{0+0} = \dfrac{0}{0}$

نثبت قيمة x

$\lim_{(0,y)} \dfrac{0-y^2}{0+y^2} = \dfrac{-\cancel{y^2}}{\cancel{y^2}} = -1$

limit does not exist

$\lim_{(x,0)} \dfrac{x^2-0}{x^2+0} = \dfrac{x^2}{x^2} = 1$



Polar تحويلات العمدات القطبية

$$\boxed{x = r\cos x \qquad y = r\sin\theta \qquad x^2+y^2 = r^2}$$

$$\lim_{(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(0,0)} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\lim \frac{\cancel{r^2}\left(\cos^2\theta - \sin^2\theta\right)}{\cancel{r^2}}$$

$$\lim_{r\to 0} \cos 2\theta = \cos 2\theta \rightarrow$$

عنثابت النهاية غير موجوده

EXAMPLE 3 Evaluate the following limits if they exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{3x^2+3y^2}$ and (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin 3x}{x} = 3$$

$$\lim \frac{\sin 4x}{5x} = \frac{4}{5}$$

a) $\displaystyle\lim_{(0,0)} \frac{\sin(x^2+y^2)}{3(x^2+y^2)} = \frac{0}{0}$

$$\lim_{(0,0)} \frac{1}{3}\,\frac{\sin \cancel{r^2}}{\cancel{r^2}} = \frac{1}{3}$$

b) $\displaystyle\lim_{(0,0)} \frac{xy}{x^2+y^2}$ التعويض $\frac{0}{0}$

$r^2 = x^2+y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$

$$\lim_{r\to 0} \frac{\cancel{r^2}\cos\theta\sin\theta}{\cancel{r^2}} = \sin\theta\cos\theta$$

does not exist

Continuity at point

تكون الداله متصله اذا
كانت النهايه موجوده عند
هذه النقطه

هل الدول التاليه متصلة

$f(x,y) = x + 4y^2$ نعم   دائما النهايه موجوده
متصله دائما

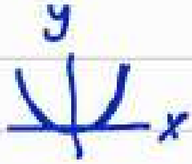
$f(x,y) = \frac{x^2+5y}{x+3}$ نعم   تكون غير متصله
عند اصفار

EXAMPLE 4 Describe the points (x, y) for which the following functions are continuous.

(a) $H(x, y) = \dfrac{2x + 3y}{y - 4x^2}$,   (b) $F(x, y) = \cos(x^3 - 4xy + y^2)$

a) the function is continous at every point tha $y - 4x^2 \neq 0$

$y - 4x^2 = 0$ $y = 4x^2$



الداله متصله عند جميع نقاط x و y ماعدا
النقاط التي تقع على منحنى $y = 4x^2$

Continous at all (x,y) exept the point along parabola $y = 4x^2$

b) $F(x,y) = \cos(\underbrace{x^3 - 4xy + y^2}_{\text{polynomial}})$

$= \cos(r)$

متصله على جميع الارقام

Continous for all (x,y)

Problem Set 12.3

In Problems 1–16, find the indicated limit or state that it does not exist.

1. $\lim_{(x,y)\to(1,3)} (3x^2y - xy^3)$

$= 3(1)^2(3) - 1(27) = -18$

3. $\lim_{(x,y)\to(2,\pi)} [x\cos^2(xy) - \sin(xy/3)]$

$2\cos^2(2\pi) - \sin\frac{2\pi}{3}$

$2 - \frac{\sqrt{3}}{2} = 1.13$

5. $\lim_{(x,y)\to(-1,2)} \frac{xy - y^3}{(x + y + 1)^2}$

$= \frac{-2-8}{(-1+2+1)^2} = \frac{-10}{4} = \frac{-5}{2}$

7. $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \frac{\sin 0}{0} = \frac{0}{0}$ لاينفع التعويض

$\lim \frac{\sin a}{a} = 1$

9. $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{x^4-y^4}$

$$= \frac{0+0}{0-0} = \frac{0}{0}$$ لا ينفع التعويض

$$\lim_{(0,0)} \frac{\cancel{x^2+y^2}}{\cancel{(x^2+y^2)}(x^2-y^2)} = \lim_{(0,0)} \frac{1}{x^2-y^2} = \frac{1}{0}$$

غير موجودة

11. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$ $= \dfrac{0}{\sqrt{0+0}} = \dfrac{0}{0}$

$$\lim_{r\to 0} \frac{r\cos\theta \; r\sin\theta}{\sqrt{r^2}} = \lim_{r\to 0} r\sin\theta\cos\theta$$

$$= 0\sin\theta\cos\theta = 0$$

موجوده = صفر

13. $\lim_{(x,y)\to(0,0)} \dfrac{x^{7/3}}{x^2+y^2}$

$$\lim_{r\to 0} \frac{(r\cos\theta)^{7/3}}{r^2} = \lim_{r\to 0} \frac{r^{7/3}}{r^2}\cos\theta$$

$$\lim_{r\to 0} r^{1/3}\cos\theta = 0^{1/3}\cos\theta = 0$$

15. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^4} = \dfrac{0}{0+0} = \dfrac{0}{0}$

$$\lim_{r\to 0} \frac{r^2\cos^2\theta \; r^2\sin^2\theta}{r^2\cos^2\theta + r^4\sin^4\theta}$$

$$\lim_{r\to 0} \left[\frac{r^2\cos^2\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}\right]$$

$r=0$

$\dfrac{0}{\cos^2\theta} = 0$

$\cos^2\theta = 0$

$\dfrac{0}{0 + r^2\sin^4\theta} = 0$

In Problems 17–26, describe the largest set S on which it is correct to say that f is continuous.

17. $f(x, y) = \dfrac{x^2 + xy - 5}{x^2 + y^2 + 1}$

موجب موجب

continous for all (x,y)

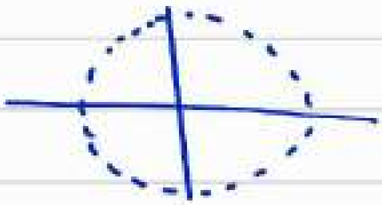
$x^2+y^2+1 \neq 0$

19. $f(x, y) = \ln(1 - x^2 - y^2)$

$1 - x^2 - y^2 > 0$

دائرة نصف قطرها اقل من 1 $\quad x^2 + y^2 < 1$

continous inside unit circle

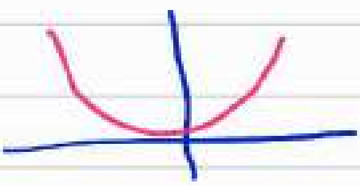


21. $f(x, y) = \dfrac{x^2 + 3xy + y^2}{y - x^2}$

$y - x^2 \neq 0 \qquad y \neq x^2$

متصلة على كل الارقام ما عدا التي تقع

على مسار $y = x^2$



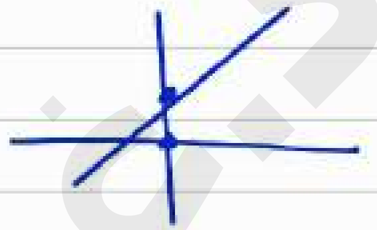
the entire plane except $y=x^2$

23. $f(x, y) = \sqrt{x - y + 1}$

$x - y + 1 \geq$ $\qquad$ $x + 1 \geq y$

$y \leq x + 1$

متصله من كل النقاط الي تحت
الخط $y = x + 1$



continous on region below
$y = x + 1$

25. $f(x, y, z) = \dfrac{1 + x^2}{x^2 + y^2 + z^2}$ $\qquad$ $x^2 + y^2 + z^2 \neq 0$

لاموجب لا موجب لا موجب

غير متصل عندما
$(0, 0, 0)$

continous for all (x, y, z)
since $(x, y, z) \neq 0$

33. Let

$$f(x, y) = \begin{cases} \dfrac{x^2 - 4y^2}{x - 2y}, & \text{if } x \neq 2y \\ g(x), & \text{if } x = 2y \end{cases}$$

If f is continuous in the whole plane, find a formula for $g(x)$.

① النهاية يجب ان تكون متساوية من الطرفين

② النهاية تساوي $f(x)$ عند نقطة التوقف

③ الدالة متصلة عند كل النقط

$x - 2y \neq 0 \qquad x \neq 2y$

$$\lim_{x \to 2y} \frac{x^2 - 4y^2}{x - 2y} = g(x)$$

$$\lim_{x \to 2y} \frac{(x + 2y)\cancel{(x - 2y)}}{\cancel{x - 2y}}$$

$$2y + 2y = 4y$$

$$g(x) = 2x$$

35. Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

does not exist by considering one path to the origin along the x-axis and another path along the line $y = x$.

x axis $\quad y=0$

$$\lim_{(x,0)} \frac{0}{x^2+0} = 0$$

does not exist

$$\lim_{(x,x)} \frac{x^2}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

36. Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy + y^3}{x^2 + y^2}$$

does not exsit

$y=0$ $\quad \lim_{(x,0)} \frac{0+0}{x^2+0} = 0$

$y=x$ $\quad \lim_{(x,x)} \frac{x^2+x^3}{x^2+x^2} = \frac{\cancel{x^2}(1+x)}{2\cancel{x^2}}$

$$\lim_{x\to 0} \frac{1+x}{2} = \frac{1}{2}$$

12.4
Differentiability

قابلية الاشتقاق

* اذا كانت الداله متصلة اذا هي قابله للاشتقاق

$f(x) = x^2+4$ متصله قابله للاشتقاق

$f'(x) = 2x$

$f(x) = x^2 + 2xy$ متصلة وقابله للاشتقاق

Gradient : $\nabla f \Rightarrow$ مشتقه الداله f بجميع الاتجاهات

$$\boxed{\nabla f = f_x i + f_y j + f_z k}$$

مثال

$f(x) = x^2 + 2xy$ $\quad \nabla f$

$f_x = 2x + 2y \qquad f_y = 2x$

$\nabla f = (2x+2y)i + (2x)j$

EXAMPLE 1 Show that $f(x, y) = xe^y + x^2y$ is differentiable everywhere and calculate its gradient. Then find the equation of the tangent plane at $(2, 0)$.

الداله قابله للاشتقاق لانها متصله

$f(x,y)$ is differentiable everywhere because it is continous

$$\nabla f = (\ f_x\ )i + (f_x\ )j$$

$$\nabla f = (e^y + 2xy)i + (xe^y + x^2)j$$

$$\nabla f(2,0) = (1)i + (2+4)j$$

$$= i + 6j = \langle 1, 6 \rangle$$

EXAMPLE 2 For $f(x, y, z) = x \sin z + x^2y$, find $\nabla f(1, 2, 0)$.

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$\nabla f = (\sin z + 2xy)i + (x^2)j + (x\cos z)k$$

$$\nabla f(1,2,0) = (0+4)i + 1j + 1k$$

$$= 4i + j + k = \langle 4, 1, 1 \rangle$$

Problem Set 12.4

In Problems 1–10, find the gradient ∇f.

1. $f(x, y) = x^2y + 3xy$ **2.** $f(x, y) = x^3y - y^3$

1) $\nabla f = f_x i + f_y j + f_z k$

$\nabla f = (2xy + 3y)i + (x^2 + 3x)j$

$\langle 2xy + 3y, x^2 + 3x \rangle$

2) $\nabla f = f_x i + f_y j$

$= (3x^2y)i + (x^3 - 3y^2)j$

$\langle 3x^2y, x^3 - 3y^2 \rangle$

In Problems 11–14, find the gradient vector of the given function at the given point **p**. *Then find the equation of the tangent plane at* **p** *(see Example 1).*

11. $f(x, y) = x^2y - xy^2$, $\mathbf{p} = (-2, 3)$

$\nabla f = f_x i + f_y j$

$= (2xy - y^2)i + (x^2 - 2xy)j$

$= (-12 - 9)i + (4 + 12)j = -21i + 16j$

$\langle -21, 16 \rangle$

12.5
Directional Derivatives and Gradients

Gradiant vector

$\boxed{\nabla f}$

$f(x) = x^2 + y^2$

$\nabla f(x)$

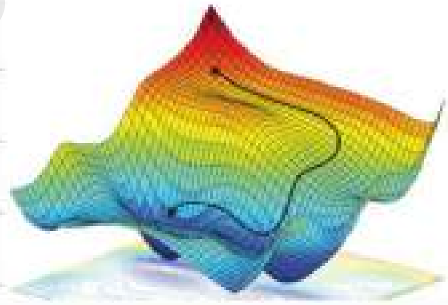
$f_x = 2x \qquad f_y = 2y$

$\nabla f = 2xi + 2yj$

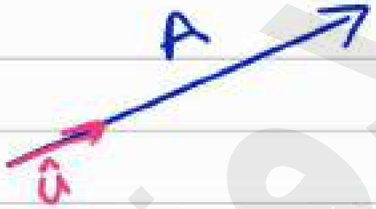
$f(x, y, z)$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

متجه يعبر عن كل المشتقات الجزئيه



$\hat{u}$ unit vector متجه الوحده

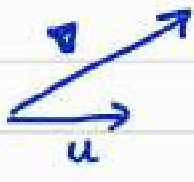


$$\hat{u} = \frac{\vec{A}}{\|A\|}$$

مثال اوجد متجه الوحده للمتجه $A = 4i + 5j$

$$\hat{u} = \frac{4i + 5j}{\sqrt{16+25}} = \frac{4i}{\sqrt{41}} + \frac{5j}{\sqrt{41}} = \left\langle \frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}} \right\rangle$$

Directional Derivative $D_u f$



معدل تغير الداله f باتجاه متجه الوحده $\hat{u}$

$$\boxed{D_u f(P) = \hat{u} \cdot \nabla f(P)}$$

الجواب
رحم

EXAMPLE 1 If $f(x, y) = 4x^2 - xy + 3y^2$, find the directional derivative ($D_u f$) of f at $(2, -1)$ (P) in the direction of the vector $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ (u).

$f(x,y) = 4x^2 - xy + 3y^2 \qquad P(2,-1)$

① حساب متجه الوحده

$a = 4i + 3j$

$u = \dfrac{4i+3j}{\sqrt{25}} = \dfrac{4}{5}i + \dfrac{3}{5}j = \langle \frac{4}{5}, \frac{3}{5} \rangle$

② حساب ∇f

$f_x = 8x - y$

$f_y = -x + 6y$

$\nabla f = f_x i + f_y j = (8x-y)i + (-x+6y)j$

$\nabla f = (8x-y)i + (-x+6y)j$

③ تعويض النقطه P في ∇f $(2,-1)$

$\nabla f(P) = 17i - 8j = \langle 17, -8 \rangle$

④ نطبق قانون $D_u \nabla f(P)$

$D_u \nabla f(P) = \hat{u} \cdot \nabla f(P)$

$= \langle \frac{4}{5}, \frac{3}{5} \rangle \cdot \langle 17, -8 \rangle = \frac{4}{5}(17) + \frac{3}{5}(-8) = \frac{44}{5}$

EXAMPLE 2 Find the directional derivative of the function $\overset{f}{f(x, y, z)} =$ $xy \sin z$ at the point $(1, 2, \pi/2)$ in the direction of the vector $\mathbf{a} = \underbrace{\mathbf{i} + 2\mathbf{j}}_{u} + 2\mathbf{k}$.

$$\hat{u} = \frac{i + 2j + 2k}{\sqrt{1+4+4}} = \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$$

$$f(x) = xy\sin z \quad \begin{cases} f_x = y\sin z \ i \\ f_y = x\sin z \ j \\ f_z = xy\cos z \ k \end{cases}$$

$$\nabla f = y\sin z\, i + x\sin z\, j + xy\cos z\, k$$

$$\nabla f(1, 2, \tfrac{\pi}{2}) = 2\sin\tfrac{\pi}{2} i + 1\sin\tfrac{\pi}{2} j + 2\cos\tfrac{\pi}{2} k$$

$$\nabla f(P) = 2i + j + 0k = \langle 2, 1, 0 \rangle$$

$$D_u f(1, 2, \pi/2) = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle \cdot \langle 2, 1, 0 \rangle$$

$$\frac{1}{3}(2) + \frac{2}{3}(1) + \frac{2}{3}(0)$$

$$\frac{2}{3} + \frac{2}{3} + 0 = \frac{4}{3}$$

EXAMPLE 3 Suppose that a bug is located on the hyperbolic paraboloid $z = y^2 - x^2$ at the point $(1, 1, 0)$, as in Figure 2. In what direction should it move for the steepest climb and what is the slope as it starts out?

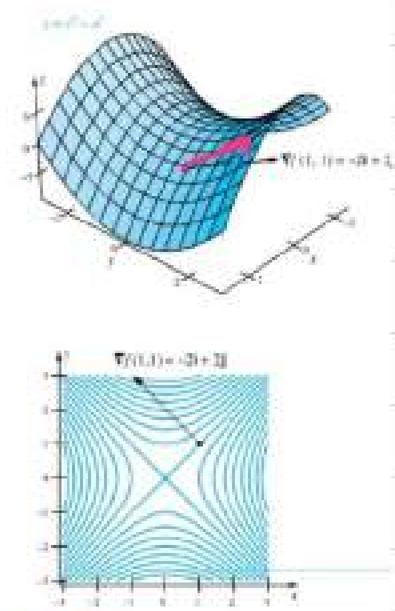


Figure 2

$\nabla f \Rightarrow$ يمثل التغير الاكبر

او الاتجاه الذي يجب ان نسلكه للحصول

على اكبر معدل زياده في الاتجاه

$f_x = -2x \qquad f_y = 2y$

$\nabla f = -2x\,i + 2y\,j = \langle -2x, 2y \rangle$

$\nabla f(1,1,0) = -2(1)i + 2(1)j = \boxed{-2i + 2j}$

$\langle -2, 2 \rangle$

الميل هو طول متجه ∇f

$\|\nabla f\| = \sqrt{2^2 + 2^2} = \sqrt{8}$

$\nabla f(p)$ increases most rapidly at point p in the diretion of ∇f

$-\nabla f(p)$ decrease most rapidly in opposite diretion

EXAMPLE 5 If the temperature at any point in a homogeneous body is given by $\boxed{T = e^{xy} - xy^2 - x^2yz}$, what is the direction of the greatest drop in temperature at the point $(1, -1, 2)$?

$T_x = y e^{xy} - y^2 - 2xyz$

$T_y = x e^{xy} - 2yx - x^2 z$

$$T_z = -x^2 y$$

$$\nabla T = (ye^{xy} - y^2 - 2xyz)i + (xe^{xy} - 2yx - x^2 z)j - x^2 yzk$$

$$\nabla T(1,-1,2) = (-e^{-1} - 1 + 4)i + (e^{-1} + 2 - 2)j + k$$

$$\nabla T(1,-1,2) = (-e^{-1} + 3)i + e^{-1}j + k$$

$$-\nabla T(1,-1,2) = (e^{-1} - 3)i - e^{-1}j - k$$

Problem Set 12.5

In Problems 1–8, find the directional derivative of f at the point **p** *in the direction of* **a**.

1. $f(x, y) = x^2y$; $\mathbf{p} = (1, 2)$; $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$

$$\hat{u} = \frac{3i - 4j}{\sqrt{25}} = \frac{3}{5}i - \frac{4}{5}j = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$$

$$f_x = 2xy \qquad f_y = x^2$$

$$\nabla f = 2xy\,i + x^2 j = \langle 2xy, x^2 \rangle$$

$$\nabla f(1,2) = \langle 4, 1 \rangle$$

$$D_u f(1,2) = \hat{u} \cdot \nabla f(p)$$

$$= \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle \langle 4, 1 \rangle = \frac{3}{5} \cdot 4 + \frac{-4}{5} \cdot 1$$

$$= \frac{12}{5} - \frac{4}{5} = \frac{8}{5}$$

3. $f(x, y) = 2x^2 + xy - y^2$; $\mathbf{p} = (3, -2)$; $\mathbf{a} = \mathbf{i} - \mathbf{j}$

$$\hat{u} = \frac{i - j}{\sqrt{2}} = \frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}}j = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$f_x = 4x + y \qquad f_y = x - 2y$$

$$\nabla f = (4x + y)\,i + (x - 2y)\,j$$

$$\nabla f(P) = 10i + 7j = \langle 10, 7 \rangle$$

$$D_u f(3,2) = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \langle 10, 7 \rangle$$

$$\frac{10}{\sqrt{2}} - \frac{7}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

5. $f(x, y) = e^x \sin y$; $\mathbf{p} = (0, \pi/4)$; $\mathbf{a} = \mathbf{i} + \sqrt{3}\mathbf{j}$

$$\hat{u} = \frac{i + \sqrt{3}j}{\sqrt{1+3}} = \frac{1}{2}i + \frac{\sqrt{3}}{2}j = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$f_x = e^x \sin y \qquad f_y = e^x \cos y$$

$$\nabla f = e^x \sin y \, i + e^x \cos y \, j$$

$$\nabla f\left(0, \frac{\pi}{4}\right) = e^0 \sin\frac{\pi}{4} i + e^0 \cos\frac{\pi}{4} j = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$D_u f\left(0, \frac{\pi}{4}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

7. $f(x, y, z) = x^3 y - y^2 z^2$; $\mathbf{p} = (-2, 1, 3)$; $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

$$\hat{u} = \frac{i - 2j + 2k}{\sqrt{1+4+4}} = \frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k = \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$$

$$f_x = 3x^2 y \qquad f_y = x^3 - 2yz^2 \qquad f_z = -2y^2 z$$

$\nabla f = 3x^2y\, i + (x^3 - 2yz^2)\, j - 2y^2z\, k$

$\nabla f(-2,1,3) = 12i + (-8-18)j - 6k$

$= 12i - 26j - 6k = \langle 12, -26, -6 \rangle$

$D_u f(p) = \langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \rangle \cdot \langle 12, -26, -6 \rangle$

$\frac{1}{3}(12) + \frac{52}{3} - \frac{12}{3} = \cancel{4} + \frac{52}{3} - 4 = \frac{52}{3}$

In Problems 9–12, find a unit vector in the direction in which f increases most rapidly at **p**. *What is the rate of change in this direction?*

9. $f(x, y) = x^3 - y^5$; $\mathbf{p} = (2, -1)$

$\nabla f = f_x i + f_y j = 3x^2 i - 5y^4 j = \langle 3x^2, -5y^4 \rangle$

$\nabla f(2,-1) = 12i - 5j$

$\vec{u} = \frac{12i - 5j}{\sqrt{12^2 + 5^2}} = \frac{12}{13} i - \frac{5}{13} j$

11. $f(x, y, z) = x^2yz;\ \mathbf{p} = (1, -1, 2)$

$$\nabla f = f_x\, i + f_y\, j + f_z\, k$$

$$= 2xyz\, i + x^2z\, j + x^2y\, k$$

$$\nabla f(1,-1,2) = -4i + 2j - k$$

$$\hat{u} = \frac{-4i + 2j - k}{\sqrt{4^2 + 2^2 + 1^2}} = \frac{-4}{\sqrt{21}} i + \frac{2}{\sqrt{21}} j - \frac{k}{\sqrt{21}}$$

12. $f(x, y, z) = xe^{yz};\ \mathbf{p} = (2, 0, -4)$

$$\nabla f = f_x\, i + f_y\, j + f_z\, k$$

$$= e^{yz}\, i + xze^{yz}\, j + xye^{yz}$$

$$\nabla f(2,0,-4) = i - 8j$$

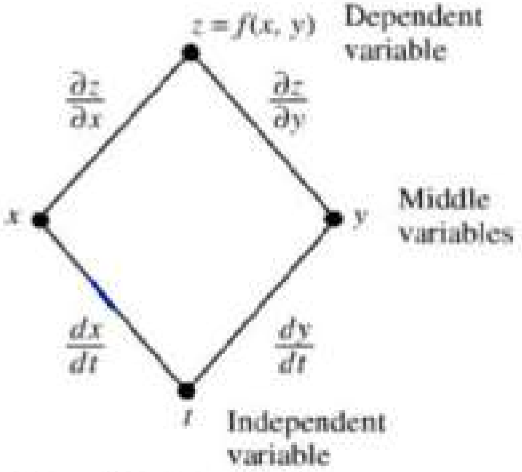
$$\hat{u} = \frac{i - 8j}{\sqrt{1^2 + 8^2}} = \frac{i - 8j}{\sqrt{65}}$$

12.6
The Chain Rule

$y = x(t)$ $\qquad \dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

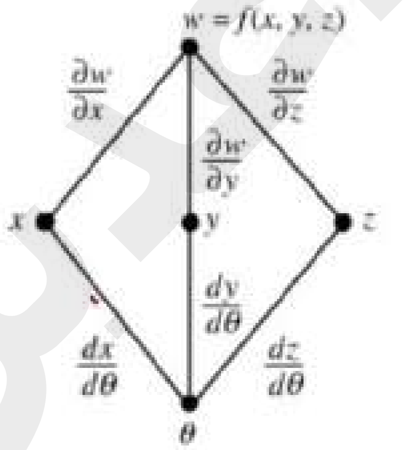
The Chain Rule: Two-Variable Case

Here is a device that may help you to remember the Chain Rule.



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

The Chain Rule: Three-Variable Case



$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{dx}{d\theta} + \frac{\partial w}{\partial y}\frac{dy}{d\theta} + \frac{\partial w}{\partial z}\frac{dz}{d\theta}$$

$z = f(x, y)$ → $x(t)$, $y(t)$

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$w = f(x, y, z)$ → $x(t)$, $y(t)$, $z(t)$

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} + \frac{dw}{dy}\frac{dy}{dt} + \frac{dw}{dx}\frac{dx}{dt}$$

EXAMPLE 1 Suppose that $z = x^3y$, where $x = 2t$ and $y = t^2$. Find dz/dt.

$$\frac{dz}{dt} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt}$$

$$3x^2y \quad 2 \quad x^3 \quad 2t$$

$$6x^2y + 2x^3t$$

$$6(2t)^2(t^2) + 2(2t)^3t = 24t^4 + 16t^4 = 40t^4$$

$$z = x^3y$$

$$z = 8t^3t^2$$

$$z = 8t^5$$

$$\frac{dz}{dt} = 40t^4$$

EXAMPLE 3 Suppose that $w = x^2y + y + xz$, where $x = \cos\theta$, $y = \sin\theta$, and $z = \theta^2$. Find $dw/d\theta$ and evaluate it at $\theta = \pi/3$.

$$\frac{dw}{d\theta} = \frac{dw}{dx}\frac{dx}{d\theta} + \frac{dw}{dy}\frac{dy}{d\theta} + \frac{dw}{dz}\frac{dz}{d\theta}$$

$$= (2xy+z)(-\sin\theta) + (x^2+1)(\cos\theta) + (x)(2\theta)$$

$$= (2\cos\theta\sin\theta + \theta^2)(-\sin\theta) + (\cos^2\theta + 1)\cos\theta + 2\theta\cos\theta$$

$$= -2\sin\theta\cos\theta - \theta^2\sin\theta + \cos^3\theta + \cos\theta + 2\theta\cos\theta$$

EXAMPLE 4 If $z = 3x^2 - y^2$, where $x = 2s + 7t$ and $y = 5st$, find $\partial z/\partial t$ and express it in terms of s and t.

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$6x \qquad 7 \qquad 2y \qquad 5s$$

$$= 42x + 10ys$$

$$= 42(2s+7t) + 10(5st)s$$

$$= 84s + 294 + 50s^2t$$

EXAMPLE 5 If $w = x^2 + y^2 + z^2 + xy$, where $x = st$, $y = s - t$, and $z = s + 2t$, find $\partial w/\partial t$.

$$\frac{\partial w}{\partial t} = \frac{dw}{dx} \cdot \frac{dx}{dt} + \frac{dw}{dy} \cdot \frac{dy}{dt} + \frac{dw}{dz} \frac{dz}{dt}$$

$$= (2x+y) \cdot s + (2y+x)(-1) + (2z)(2)$$

$$= 2xs + ys - 2y - x + 4z$$

$$= 2(st)s + (s-t)s - 2(s-t) - (st) + 4(s+2t)$$

$$= 2s^2t + s^2 - st - 2s - \underline{2t} + st + 4s \underline{+8t}$$

$$= 2s^2t + s^2 + 2s - 2st + 10t$$

Implicit Differentiation

$$3y = 4x + 2xy$$

$$3\frac{dy}{dx} = 4 + 2x\frac{dy}{dx} + 2y$$

طريقة الاشتقاق الضمني

$$3\frac{dy}{dx} - 2x\frac{dy}{dx} = 4 + 2y$$

$$\frac{dy}{dx}(3-2x) = 4 + 2y$$

$$\frac{dy}{dx} = \frac{4+2y}{3-2x}$$

$$\boxed{\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}}$$

$$3y = 4x + 2xy \qquad \underbrace{3y - 4x - 2xy}_{F} = 0$$

$$\frac{dy}{dx} = -\frac{-4-2y}{3-2x} = \frac{4-2y}{3-2x}$$

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

EXAMPLE 6 Find dy/dx if $x^3 + x^2y - 10y^4 = 0$ using

(a) the Chain Rule, and (b) implicit differentiation.

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

اشتق كل حد ، عند اشتقاق y اضرب بـ $\frac{dy}{dx}$

a)

$$F = x^3 + x^2y - 10y^4$$

$$\frac{dy}{dx} = -\frac{3x^2 + 2xy}{x^2 - 40y^3}$$

b)

$$x^3 + x^2y - 10y^4 = 0$$

$$3x^2 + 2xy + x^2\frac{dy}{dx} - 40y^3\frac{dy}{dx} = 0$$

$$40y^3\frac{dy}{dx} - x^2\frac{dy}{dx} = 3x^2 + 2xy$$

$$\frac{dy}{dx}\left(40y^3 - x^2\right) = 3x^2 + 2xy$$

$$\frac{dy}{dx} = -\frac{3x^2 + 2xy}{-40y^3 + x^3}$$

EXAMPLE 7 If $F(x, y, z) = x^3 e^{y+z} - y\sin(x - z) = 0$ defines z implicitly as a function of x and y, find $\partial z/\partial x$.

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

$$= -\frac{3x^2 e^{yz} - y\cos(x-z)}{x^3 e^{y+z} + y\cos(x-z)}$$

Problem Set 12.6

In Problems 1–6, find dw/dt by using the Chain Rule. Express your final answer in terms of t.

1. $w = x^2y^3$; $x = t^3$, $y = t^2$

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} + \frac{dw}{dy} \cdot \frac{dy}{dt}$$

$$= 2xy^3(3t^2) + 3x^2y^2(2t)$$

$$= 2(t^3)(t^2)^3(3t^2) + 3(t^3)^2(t^2)^2(2t)$$

$$6t^{11} + 6t^{11} = 12t^{11}$$

3. $w = e^x \sin y + e^y \sin x$; $x = 3t$, $y = 2t$

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt} + \frac{dw}{dy} \cdot \frac{dy}{dt}$$

$$= (e^x \sin y + e^y \cos x)3 + (e^x \cos y + e^y \sin x)2$$

$$= 3e^{3t} \sin 2t + 3e^{2t} \cos 3t + 2e^{3t} \cos 2t + 2e^{2t} \sin 3t$$

5. $w = \sin(xyz^2);\ x = t^3,\ y = t^2,\ z = t$

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\cdot\frac{dy}{dt} + \frac{dw}{dz}\cdot\frac{dz}{dt}$$

$$\frac{dw}{dt} = \cos(xyz^2)(yz^2)(3t^2) + \cos(xyz^2)(xz^2)(2t)$$
$$+ \cos(xyz^2)(2xyz)\,1$$

$$= \cos(xyz^2)\left[3yz^2t^2 + 2t\,xz^2 + 2xyz\right]$$
$$= \cos(t^7)\left[3t^2t^2t^2 + 2t\,t^3t^2 + 2t^3t^2t\right]$$
$$= \cos(t^7)\left[7t^6\right]$$

In Problems 7–12, find $\partial w/\partial t$ by using the Chain Rule. Express your final answer in terms of s and t.

7. $w = x^2y;\ x = st,\ y = s - t$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= (2xy)(s) + x^2(-1)$$

$$= 2(st)(s-t)(s) - s^2t^2$$

$$s^3t - 2s^2t^2 - s^2t^2$$
$$s^3t - 3s^2t^2 = s^2t\left[1 - 3t\right]$$

9. $w = e^{x^2+y^2}$; $x = s\sin t$, $y = t\sin s$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= 2x\,e^{x^2+y^2}\cdot s\cos t + 2y\,e^{x^2+y^2}\cdot \sin s$$

$$2\,e^{x^2+y^2}\,[x s\cos t + y\sin s]$$

$$= 2\,e^{s^2\sin^2 t + t^2\sin^2 s}\,[s^2\sin t\cos t + t^2\sin^2 s]$$

$$= 2ts\sin t\cos t + 2t\sin^2 s\,(e^{s^2\sin^2 t + t^2\sin^2 s})$$

11. $w = \sqrt{x^2+y^2+z^2}$; $x = \cos st$, $y = \sin st$, $z = s^2 t$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial t} = \frac{2x}{2\sqrt{x^2+y^2+w^2}}(-\sin st)\cdot s + \frac{2y}{2\sqrt{x^2+y^2+z^2}}\cdot\cos st\,(s)$$

$$+ \frac{2z}{2\sqrt{x^2+y^2+z^2}}\cdot s^2$$

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\left[-xs\sin st + ys\cos(st) + zs^2\right]$$

$$\frac{1}{\sqrt{\sin^2 st+\cos^2 st+s^4t^2}}\left[-s\cos st\sin st + s\sin st\cos st + s^4t\right]$$

$$\frac{1}{\sqrt{1+s^4t^2}}\left[s^4t\right]$$

13. If $z = x^2y$, $x = 2t + s$, and $y = 1 - st^2$, find

$$\left.\frac{\partial z}{\partial t}\right|_{s=1,\,t=-2}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$= 2xy(2) + x^2(-2st)$$

$$= 2(2t+s)(1-st^2)(2) + (2t+s)^2(-2st)$$

$$= 2(-4+1)(1-4)(2) + (-4+1)^2(4)$$

$$36 + 36 = 72$$

15. If $w = u^2 - u\tan v$, $u = x$, and $v = \pi x$, find

$$\left.\frac{dw}{dx}\right|_{x=1/4}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{du}{dt} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$= (2u - \tan v)(1) - u\sec^2 v\,(\pi)$$

$$= (2x - \tan\pi x) - x\pi\sec^2\pi x$$

$$= \left(\frac{1}{2} - 1\right) - \frac{\pi}{4}\left(\frac{2}{\sqrt{2}}\right)^2$$

$$-\frac{1}{2} - \frac{\pi}{2} = -\frac{1+\pi}{2}$$

16. If $w = x^2y + z^2$, $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$, find

$$\left.\frac{\partial w}{\partial \theta}\right|_{\rho=2,\theta=\pi,\phi=\pi/2}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$= (2xy)(-\rho\sin\theta\sin\phi) + (x^2)(\rho\cos\theta\sin\phi) + (2z)(0)$$

$$= (2\rho\cos\theta\sin\phi)(-\rho^2\sin^2\theta\sin^2\phi)$$

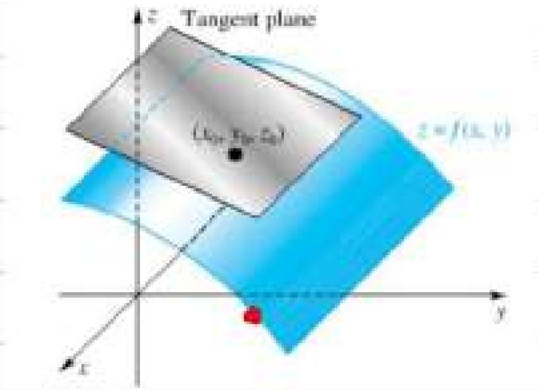
$$+ \rho^3\cos^2\theta\sin^2\phi$$

$$= 4\left(\cos\pi\sin\frac{\pi}{2}\right)\left(2\sin\pi\sin\frac{\pi}{2}\right)$$

$$+ 2^3\cos^3\pi\sin\frac{\pi}{2} + 4\cos\frac{\pi}{2}$$

$$0 - 8 = -8$$

12.7
Tangent Planes and Approximations



معادلة السطح المماسي عند نقطة
(y_0, x_0, z_0)

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE 1 Find the equation of the tangent plane (Figure 3) to $z = x^2 + y^2$ at the point $(1, 1, 2)$.

$x_0 \; y_0 \; z_0$

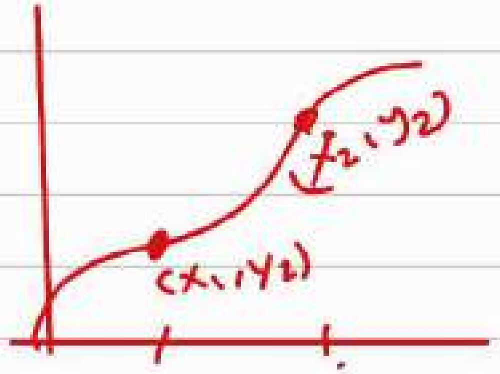
$$z - z_0 = \overset{2x}{f_x}(1,1)(x - x_0) + \overset{2y}{f_y}(1,1)(y - y_0)$$

$$z - 2 = 2(x - 1) + 2(y - 1)$$

$$z - 2 = 2x - 2 + 2y - 2$$

$$z = 2x + 2y - 2$$

$$2x + 2y - z = 2$$



$f(x, y)$

$\Delta z = f(x_2, y_1) - f(x_1, y_1)$

differential of dz

$$\boxed{df(x,y) = dz = f_x\, dx + f_y\, dy}$$

EXAMPLE 3 Let $z = f(x, y) = 2x^3 + xy - y^3$. Compute Δz and dz as (x, y) changes from $(2, 1)$ to $(2.03, 0.98)$.

$z = 2x^3 + xy - y^3$ → Δz, dz

$x_0 \quad x_1$

$(2,1) \rightarrow (2.03, 0.98)$

$\Delta x = 2.03 - 2 = 0.03$

$\Delta y = 0.98 - 1 = -0.02$

$$\Delta z = f(\underset{x}{2.03}, \underset{y}{0.9}) - f(\underset{x}{2}, \underset{y}{1})$$

$$\Delta z = \left[2(2.03)^2 + (2.03)(0.98) - (0.98)^3\right] = 0.779$$

$$dz = f_x\, \overset{\Delta x}{dx} + f_y\, \overset{\Delta y}{dy}$$

$$dz = (6x^2 + y)\Delta x + (x - 3y^2)\, dy$$ عوض في نقطة

at point $(2, 1)$

$$dz = (24 + 1)(0.03) + (2 - 3)(0.02) = 0.77$$

Problem Set 12.7

In Problems 1–8, find the equation of the tangent plane to the given surface at the indicated point.

$x_0 \quad y_0 \quad z_0$

1. $x^2 + y^2 + z^2 = 16; \left(2, 3, \sqrt{3}\right)$

$f_x \quad f_y \quad f_z$

$2x \quad 2y \quad 2z$

$4 \quad 6 \quad 2\sqrt{3}$

$$f_x(x-x_0)+f_y(y-y_0)+f_z(z-z_0)=0$$

$$0 = 4(x-2)+6(y-3)+2\sqrt{3}(z-\sqrt{3})$$

$$2(x-2)+3(y-3)+\sqrt{3}(z-\sqrt{3})=0$$

$$2x-4+3y-9+\sqrt{3}z-3=0$$

$$2x+3y+\sqrt{3}z=0$$

$x_0 \quad y_0 \quad z_0$

2. $8x^2 + y^2 + 8z^2 = 16; \left(1, 2, \sqrt{2}/2\right)$

$f_x = 16x \qquad f_y = 2y \qquad f_z = 16z$

$f_x\Big|_{(1,2,2\sqrt{2})} = 16 \qquad f_y = 4 \qquad f_z = 8\sqrt{2}$

$$f_x(x-x_0)+f_y(y-y_0+f_z(z-z_0)=0$$

$$16(x-1)+4(y-2)+8\sqrt{2}\left(z-\frac{\sqrt{2}}{2}\right)=0$$

$$4(x-1)+(y-2)+2\sqrt{2}\left(z-\frac{\sqrt{2}}{2}\right)=0$$

$$4x-4+y-2+2\sqrt{2}z-2=0$$

$$4x+y+2\sqrt{2}z=8$$

3. $x^2 - y^2 + z^2 + 1 = 0;\ (1, 3, \sqrt{7})$

$f_x = 2x$
$= 2$

$f_y = -2y$
$= -6$

$f_z = 2z$
$2\sqrt{7}$

$$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$$

$$2(x-1) - 6(y-3) + 2\sqrt{7}(z - \sqrt{7}) = 0$$

$$(x-1) - 3(y-3) + \sqrt{7}(z - \sqrt{7}) = 0$$

$$x - 1 - 3y + 9 + \sqrt{7}z - 7 = 0$$

$$x - 3y + \sqrt{7}z = -1$$

5. $z = \dfrac{x^2}{4} + \dfrac{y^2}{4};\ (2, 2, 2)$

$f_x = \dfrac{x}{2}$
$= \dfrac{2}{2} = 1$

$f_y = \dfrac{y}{2}$
$= \dfrac{2}{2} = 1$

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$

$$z - 2 = (x - 2) + (y - 2)$$

$$z = x + y - 2$$

$$x + y - z = 2$$

7. $z = 2e^{3y}\cos 2x$; $(\pi/3, 0, -1)$

$x \quad y$

$f_x = -4e^{3y}\sin 2x$

$f_y = 6e^{3y}\cos 2x$

$f_x\big|_{\pi/3,0} = -4e^{3(0)}\sin\frac{2\pi}{3}$

$= -4\frac{\sqrt{3}}{2} = -2\sqrt{3}$

$f_y = 6e^{3(0)}\cos\frac{2\pi}{3}$

$= 6\left(\frac{-1}{2}\right) = -3$

$$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$$

$$z + 1 = -2\sqrt{3}\left(x - \frac{\pi}{3}\right) - 3(y - 0)$$

$$z = -2\sqrt{3}x + 2\frac{\sqrt{3}\pi}{3} - 3y - 1$$

C *In Problems 9–12, use the total differential dz to approximate the change in z as (x, y) moves from P to Q. Then use a calculator to find the corresponding exact change Δz (to the accuracy of your calculator). See Example 3.*

9. $z = 2x^2y^3$; $P(1, 1)$, $Q(0.99, 1.02)$

$2(0.99)^2(1.02)^3 - 2 = \square$

0.08

$0.99 - 1 = -0.01$

$$dz = f_x\,dx + f_y\,dy$$

$$= 4xy^3\,dx + 6x^2y^2\,dy$$

$$= 4(1)(1)^3(-0.01) + 6(1)^2(1)^2(0.02)$$

$$-0.04 + 0.12 = 0.08$$

11. $z = \ln(x^2y)$; $P(-2, 4)$, $Q(-1.98, 3.96)$

$$dx = -1.98 - -2 = 0.02$$

$$dy = 3.96 - 4 = -0.04$$

$$dZ = f_{x_0}\,dx + f_{y_0}\,dy$$

$$= \frac{2xy}{x^2y}\,dx + \frac{x^2}{x^2y}\,dy$$

$$= \frac{2}{x_0}\,dx + \frac{1}{y_0}\,dy$$

$$= \frac{2}{-2}(0.02) + \frac{1}{4}(-0.04) = -0.02 - 0.01 = -0.03$$

12. $z = \tan^{-1} xy$; $P(-2, -0.5)$, $Q(-2.03, -0.51)$

$$-2.03 - -2 = -0.03$$

$$-0.51 - -0.5 = -0.01$$

$$dZ = f_x\,dx + f_y\,dy$$

$$= \frac{y}{1+x^2y^2}\,dx + \frac{x}{1+x^2y^2}\,dy$$

$$= \frac{-0.5}{1+(2\times0.5)^2}(-0.03) + \frac{-2}{1+(2\times0.5)^2}(-0.01)$$

$$= \frac{0.015}{2} + 0.01 = 0.0075 + 0.01 = 0.0175$$

13. Find all points on the surface

$z = f(x,y)$

$$z = x^2 - 2xy - y^2 - 8x + 4y$$

where the tangent plane is horizontal.

اوجد النقاط التي تكون فيها السطح المماس موازي للسطح

$f_x = 0$

$f_y = 0$

$f_x \Rightarrow \quad 2x - 2y - 8 = 0$

$f_y \Rightarrow \quad -2x - 2y + 4 = 0$

$-4y - 4 = 0$

$\boxed{y = -1}$

$\boxed{x = 3}$

$2x^2 + 3y^2 - 1 = 0$

14. Find a point on the surface $z = 2x^2 + 3y^2$ where the tangent plane is parallel to the plane $8x - 3y - z = 0$.

tangent plane is normal to the gradiant

∇ tangent plane $= \langle 4x, 6y, -1 \rangle$

∇ plane $= \langle 8, -3, -1 \rangle$

$4x = 8 \qquad 6y = -3$

$\boxed{x = 2} \qquad \boxed{y = \frac{-1}{2}}$

نعوض في معادلة لحساب z

$z = 2(2)^2 + 3(\frac{1}{2})^2 = 8.75$

12.8
Maxima and Minima

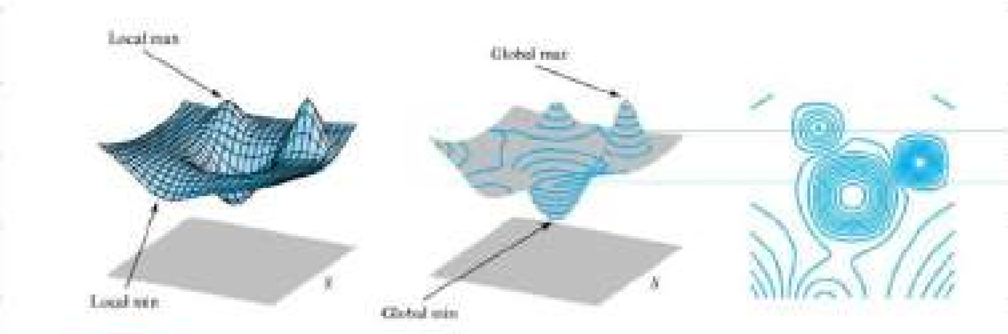
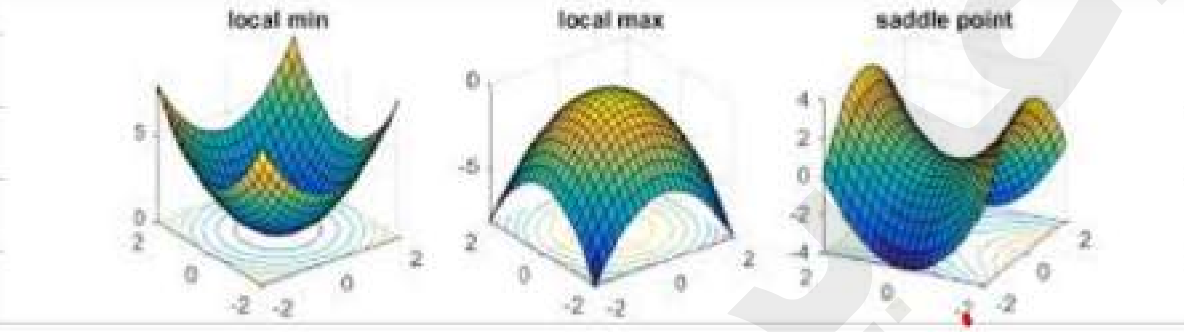


Figure 1



لتحديد القيم العظمى و الصغرى

① حساب القيم الحرجة (Critical point) $\nabla f = 0$

$f_x = 0$ $f_y = 0$ (x_0, y_0)

القيم الحرجة هي قيم x و Y التي تكون عندها المشتقة صفر

② اختبار المشتقة الثانية $D(x_0, y_0)$

$$\boxed{D = f_{xx} f_{yy} - f_{xy}^2}$$

$D>0$ موجبة و $f_{xx}<0$ سالبة Maximum

$D>0$ موجبة $f_{xx}>0$ موجبة Minimum

$D<0$ سالبة saddle point

$D=0$ هذا الاختبار غير نافع

1. Find the critical points of the function:

$$f(x, y) = x^2 - 6x + y^2 - 8y$$

$\nabla f = 0$

$f_x = 0 \quad , \quad f_y = 0$

$f_x = 2x - 6 \qquad 2x - 6 = 0 \qquad \boxed{x = 3}$

$f_y = 2y - 8 \qquad 2y - 8 = 0 \qquad \boxed{y = 4}$

$(3, 4)$

3. Find the critical points of the function. Then use the Second Derivative Test to determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point.

$$f(x, y) = x^2 - 6xy + y^2 + 4y$$

$f_x = 2x - 6y$

$f_y = -6x + 2y + 4$

$2x - 6y = 0 \qquad -6x + 2y + 4 = 0$

$x = 3y$

$-6(3y) + 2y + 4 = 0$

$-16y + 4 = 0$

$\boxed{x = \frac{3}{4}} \qquad \boxed{y = \frac{1}{4}}$

critical point

$\left(\frac{3}{4}, \frac{1}{4}\right)$

نستخدم اختبار المشتقة الثانية

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{xy} = -6$

$$D = (2)(2) - (-6)^2 = -32$$

سالبة $\quad D < 0$

saddle point

EXAMPLE 1 Find the local maximum or minimum values of $f(x, y) = x^2 - 2x + y^2/4$.

$f_x = 2x - 2$ $\qquad 2x - 2 = 0$ $\qquad \boxed{x = 1}$

$f_y = \dfrac{y}{2}$ $\qquad \dfrac{y}{2} = 0$ $\qquad \boxed{y = 0}$

Critical point $(1,0)$

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$f_{xx} = 2$ $\qquad f_{yy} = \frac{1}{2}$ $\qquad f_{xy} = 0$

$$D = 2\left(\tfrac{1}{2}\right) - 0 = 1$$

النقطه تعتبر قيمه صغرى

$(1,0)$ Global minimum

EXAMPLE 2 Find the local minimum or maximum values of $f(x, y) = -x^2/a^2 + y^2/b^2$.

$$f(x,y) = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$f_x = \frac{-2}{a^2}x \qquad \frac{-2}{a^2}x = 0 \qquad \boxed{x=0}$$

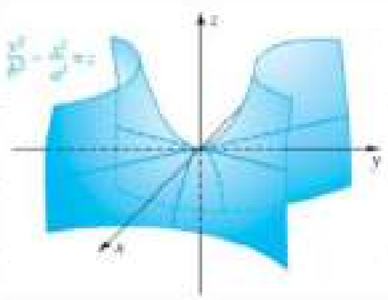
$$f_y = \frac{2}{b^2}y \qquad \frac{2}{b^2}y = 0 \qquad \boxed{y=0}$$

critical point $(0,0)$

$$D = f_{xx} f_{xy} - (f_{xy})^2$$

$$f_{xx} = -\frac{2}{a^2} \qquad f_{xy} = \frac{2}{b^2} \qquad f_{xy} = 0$$

$$D = -\frac{2}{a^2} \cdot \frac{2}{b^2} = -\frac{4}{a^2b^2}$$



$D \Rightarrow$ صالب

saddle point

EXAMPLE 3 Find the extrema, if any, of the function F defined by $F(x, y) = 3x^3 + y^2 - 9x + 4y$.

$f_x = 9x^2 - 9$ $\quad$ $9x^2 - 9 = 0$

$x^2 - 1 = 0 \begin{cases} x = +1 \\ x = -1 \end{cases}$

$f_y = 2y + 4$ $\quad$ $2y + 4 = 0$ $\quad$ $y = -2$

Critical points $\begin{cases} (1, -2) \\ (-1, 2) \end{cases}$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 18x$ $\quad$ $f_{yy} = 2$ $\quad$ $f_{xy} = 0$

$D = 36x$

---

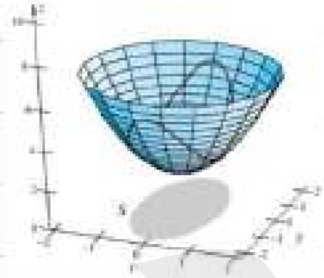
for $(1, -2)$ $\quad$ $f_{xx} = 18$ $\quad$ $D = 36$

$(1, -2)$ minimum

for $(-1, 2)$ $\quad$ $f_{xx} = -18$ $\quad$ $D = -36$

$(-1, 2)$ saddle point

EXAMPLE 5 Find the maximum and minimum values of $f(x, y) = 2 + x^2 + y^2$ on the closed and bounded set $S = \left\{(x, y): x^2 + \frac{1}{4}y^2 \le 1\right\}$.



$f_x = 2x \qquad 2x = 0 \qquad x = 0$

$f_y = 2y \qquad 2y = 0 \qquad y = 0$

critical point $(0,0)$

$D = f_{xx} f_{yy} - (f_{xy})^2$

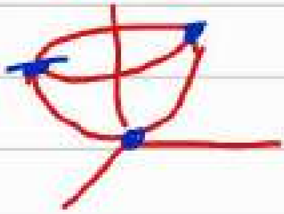
$f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{xy} = 0$

$D = 2(2) - 0 = 4$

minimum موجب f_{xx} موجب D

$(0,0)$ minimum

max أكبر min بعد حساب قيمة



$x^2 + \frac{1}{4} y^2 = 1 \qquad \sin^2 x + \cos^2 x = 1$

$\downarrow \qquad \downarrow$

$\cos^2 t \qquad (2\sin t)^2$

$g(t) = 2 + x^2 + y^2$

$$g(t) = 2 + \cos^2 t + 4\sin^2 t$$

$$g'(t) = -2\cos t \sin t + 8 \sin t \cos t$$

$$g'(t) = 6\cos t \sin t$$

$2\sin x \cos x$

$\sin 2x$

نبحث عن لنقاط الحرجة

$$3(2\cos t \sin t) = 0$$

$$3\sin 2t = 0$$

$$\sin 2t = 0$$

$\sin x = 0$

$\downarrow$

$0, \pi, 2\pi, 3\pi$

$$t = 0, \frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}$$

صار عن نقاط حرجة

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

	x	y	
t	$\cos t$	$2\sin t$	
0	1	0	3
$\pi/2$	0	2	6 ✓
π	-1	0	3
$3\pi/2$	0	-2	6 ✓
2π	1	0	3

Maximain $(0,2)$ $(0,-2)$

Problem Set 12.8

In Problems 1–10, find all critical points. Indicate whether each such point gives a local maximum or a local minimum, or whether it is a saddle point. Hint: Use Theorem C.

1. $f(x, y) = x^2 + 4y^2 - 4x$

$f_x = 2x - 4$ $\quad 2x - 4 = 0$ $\quad x = 2$

$f_y = 8y$ $\quad 8y = 0$ $\quad y = 0$

critical point $(2, 0)$

$f_{xx} = 2$ $\quad f_{yy} = 8$ $\quad f_{xy}\ 0$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= 16 - 0 = 16$

$D > 0$ $\quad f_{xx} > 0$

local minimum

3. $f(x, y) = 2x^4 - x^2 + 3y^2$

$f_x = 8x^3 - 2x$

$8x^3 - 2x = 0$

$2x(4x^2 - 1) = 0$

$2x = 0$

$x = 0$

$4x^2 - 1 = 0$

$x^2 = \frac{1}{4}$

$x = \pm \frac{1}{2}$

$f_y = 6y$ $\quad 6y = 0 \quad y = 0$

critical points $(0,0)$ $(\frac{1}{2}, 0)$ $(-\frac{1}{2}, 0)$

$f_{xx} = 24x^2 - 2 \quad f_{yy} = 6 \quad f_{xy} = 0$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

$$D = 6(24x^2 - 2)$$

for $(0,0)$ $\quad f_{xx} = -2 \quad D = -12$

saddle point

for $(\frac{1}{2}, 0)$ $\quad f_{xx} = 4 \quad D = 24$

minimum

for $(-\frac{1}{2}, 0)$ $\quad f_{xx} = 4 \quad D = 24$

minimum

5. $f(x, y) = xy$

$f_x = y \qquad y = 0$

critical point

$f_y = x \qquad x = 0$

$(0,0)$

$f_{xx} = 0 \qquad f_{yy} = 0 \qquad f_{xy} = 1$

$D = f_{xx} f_{yy} - (f_{xy})^2 = -1$

Saddle point صالبه D

7. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$

$f_x = y - \dfrac{2}{x^2} \qquad y - \dfrac{2}{x^2} = 0$

$f_y = x - \dfrac{4}{y^2} \qquad x - \dfrac{4}{y^2} = 0$

$y = \dfrac{2}{x^2}$

حل المعادلتين بالتعويض

$x - \dfrac{4}{\left(\frac{2}{x^2}\right)^2} \Rightarrow x - x^4 = 0 \qquad x = x^4 \qquad \boxed{x = 1}$

$y = \dfrac{2}{x^2} \qquad \boxed{y = 2}$

critical point (1,2)

$f_{xx} = \frac{4}{x^3}$ $f_{yy} = \frac{8}{y^3}$ $f_{xy} = 1$

$D = \frac{4 \cdot 8}{x^3 y^3} - 1$

at (1,2) $D = \frac{32}{8} - 1 = 3$

$f_{xx} = \frac{4}{1} = 4$

$D > 0$ $f_{xx} > 0$

Minimum

9. $f(x, y) = \cos x + \cos y + \cos(x + y);$
$0 < x < \pi/2, 0 < y < \pi/2$

$f_x = -\sin x - \sin(x+y)$

$f_y = -\sin y - \sin(x+y)$

$$\begin{array}{r} -\sin x - \cancel{\sin(x+y)} = 0 \\ +\sin y + \cancel{\sin(x+y)} = 0 \\ \hline -\sin x + \sin y = 0 \end{array}$$

$\sin x = \sin y$

$x = y$

لا تحقق حدود الفترات

no critical point

In Problems 11–14, find the global maximum value and global minimum value of f on S and indicate where each occurs.

11. $f(x, y) = 3x + 4y$;
$S = \{(x, y): 0 \le x \le 1, -1 \le y \le 1\}$

x	0	1
$3x$	0	3

y	-1	1
$4y$	-4	4

$$f(x,y) = 3x + 4y$$
$$= 3 + 4 = 7$$

$x=1$, $y=1$ المقصود لا هي 7

$(1,1)$ maximum value

13. $f(x, y) = x^2 - y^2 + 1$;
$S = \{(x, y): x^2 + y^2 \le 1\}$ (See Example 5.)

$f_x = 2x \qquad x = 0$

$(0,0)$

$f_y = -2y \qquad y = 0$

$f_{xx} = 2 \qquad f_{yy} = -2 \qquad f_{xy} = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = -4$ saddle point

لندرس حال القيم لتحديد
اكبر قيم عظمى او صغرى
عند الاطراف

$x^2 + y^2 = 1$

$\swarrow \qquad \searrow$

$\cos t \qquad \sin t$

$f(x,y) \Rightarrow g(t) = \underline{\cos^2 t - \sin^2 t} + 1$

$g(t) = \cos 2t + 1$

$g'(t) = -2\sin 2t$

$0 \quad \pi \quad 2\pi \quad 3\pi$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

t	$x=\cos t$	$y=\sin x$	$f(x,y)$	
0	1	0	2	max
$\pi/2$	0	1	0	min
π	-1	0	2	max
$3\pi/2$	0	-1	0	min

max $(\pm 1, 0)$ min $(0, \pm 1)$

14. $f(x, y) = x^2 - 6x + y^2 - 8y + 7$;
$S = \{(x, y): x^2 + y^2 \leq 1\}$

cos sin

$f_x = 2x - 6$ $2x - 6 = 0$ $x = 3$

$f_y = 2y - 8$ $2y - 8 = 0$ $y = 4$

critical point $(3, 4) \Rightarrow$ outside the domain

$x^2 + y^2 \leq 1$ $x^2 + y^2 = 1$

cost sint

$\Rightarrow f(x,y) = g(t) = \cos^2 t - 6\cos t + \sin^2 t$

$-8\sin t + 7$

$g(t) = 8 - 6\cos t - 8\sin t$

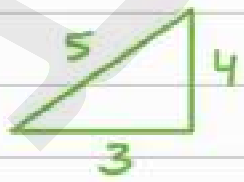
$g'(t) = 6\sin t - 8\cos t$

$6\sin t = 8\cos t$

$\frac{\sin t}{\cos t} = \frac{8}{6}$ $\quad$ $\tan t = \frac{4}{3}$

$\tan t = \frac{4}{3}$

$\sin t = \frac{4}{5}$ $\quad$ $\cos t = \frac{3}{5}$



$\sin t = \frac{4}{5}$ $\quad$ $\cos t = \frac{3}{5}$ $\quad$ $\tan t = \frac{4}{3}$

$\sin t = -\frac{4}{5}$ $\quad$ $\cos t = -\frac{3}{4}$

$g(t) = 8 - 6\cos t - 8\sin t$

$\frac{4}{5}, \frac{3}{5}$ $\quad$ $8 - 6\left(\frac{3}{5}\right) - 8\left(\frac{4}{5}\right) = -2$ $\quad$ min

$-\frac{4}{5}, -\frac{3}{5}$ $\quad$ $8 - 6\left(-\frac{3}{5}\right) - 8\left(-\frac{4}{5}\right) = 18$ $\quad$ max

Global min $\left(\frac{3}{5}, \frac{4}{5}\right)$ $\quad$ Global max $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

12.9
The Method of Lagrange Multipliers

$f(P)$ ← الدالة

$g(P)$ ← القيد

$$\boxed{\nabla f = \lambda \nabla g}$$

① نحسب ∇f

② نحسب ∇g

③ نطبق المعادلة

④ نحل المعادلة

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$f_z = \lambda g_z$$

ثم نعوض في معادلة القيد للتأكد من تحقق الحل

ثم نعوض في معادلة f للتأكد من القيم العظمى

Find the absolute extrema for

$f(x, y) = x + y$ subject to $x^2 + y^2 = 1$

$f(x, y) = x + y$ $\qquad g(x, y) = x^2 + y^2 = 1$

$$\nabla f \begin{cases} f_x = 1 \\ f_y = 1 \end{cases} \qquad \langle 1, 1 \rangle$$

$$\nabla g \begin{cases} g_x = 2x \\ g_y = 2y \end{cases} \qquad \langle 2x, 2y \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle$$

$$1 = 2\lambda x$$

$$1 = 2\lambda y$$

نقسم المعادلتين $\frac{1}{1} = \frac{2\lambda x}{2\lambda y}$

$1 = \frac{x}{y}$ $\qquad x = y$ $\qquad x^2 + y^2 = 1$

$2x^2 = 1 \qquad x = \pm\frac{1}{\sqrt{2}}$

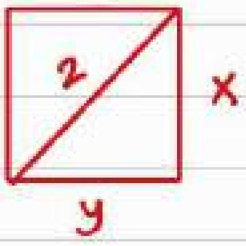
$$x = \pm\frac{1}{\sqrt{2}}$$

$x = +\frac{1}{\sqrt{2}} \qquad y = \frac{1}{\sqrt{2}} \qquad f = \frac{2}{\sqrt{2}}$ max

$x = -\frac{1}{\sqrt{2}} \qquad y = -\frac{1}{\sqrt{2}} \qquad f = -\frac{2}{\sqrt{2}}$ min

EXAMPLE 1 What is the greatest area that a rectangle can have if the length of its diagonal is 2?

ما اكبر مساحه لمستطيل اذا كان قطره يساوي 2



$$f(x,y) = xy$$

$$g(x,y) = x^2+y^2-4$$

$$x^2+y^2=2^2$$

$$\nabla f = \langle y \, , \, x \rangle$$

$$\nabla g = \langle 2x \, , \, 2y \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$$

$y = \lambda 2x \qquad x = \lambda 2y \qquad \dfrac{y}{x} = \dfrac{\cancel{2\lambda} x}{\cancel{2\lambda} y}$

$x^2 = y^2$ نعوض في معادله القيد

$$x^2+y^2=4$$

$2x^2 = 4 \qquad x^2 = 2 \qquad x = \sqrt{2} \qquad y = \sqrt{2}$

نعوض في معادله f لحساب القيم العظمى

$x = \sqrt{2} \qquad y = \sqrt{2} \qquad f = 2 \qquad (\sqrt{2}, \sqrt{2})$

EXAMPLE 2 Use Lagrange's method to find the maximum and minimum values of

$$f(x, y) = y^2 - x^2$$

on the ellipse $x^2/4 + y^2 = 1$.

حل

$f(x,y) = y^2 - x^2$ $\qquad$ $g(x,y) = x^2 + 4y^2 - 4 = 0$

$\nabla f = \langle -2x \,,\; 2y \rangle$

$\nabla g = \langle 2x \,,\; 8y \rangle$

$$\nabla f = \lambda \nabla g$$

$$\langle -2x, 2y\rangle = \lambda \langle 2x, 8y\rangle$$

$$-2x = \lambda\, 2x$$
$$2y = \lambda\, 8y$$

$$-\frac{x}{y} = \frac{x}{4y}$$

$$-4xy = xy$$

يستحيل ان تتحقق هذه المعادلة الا اذا $x=0$ و $y=0$

$x = 0$
$y = 0$

نعوضها في معادلة القيد

$\boxed{x=0}$ $\qquad$ $\cancel{x^2} + 4y^2 - 4 = 0$ $\qquad$ $y^2 = 1$ $\qquad$ $y = \pm 1$

$(0, +1)$ $\qquad$ $(0, -1)$

$\boxed{y=0}$ $\quad x^2+4y^2-\cancel{4}=0 \quad x^2=4 \quad x=\pm 2$

$(+2,0) \quad (-2,0)$

نعوض النقاط الاربعة في معادلة f

	f
$(0,1)$	1
$(0,-1)$	1
$(2,0)$	-4
$(-2,0)$	-4

$\to$ min

$\to$ max

EXAMPLE 3 Find the minimum of $f(x, y, z) = 3x + 2y + z + 5$ subject to the constraint $g(x, y, z) = 9x^2 + 4y^2 - z = 0$.

$f(x,y,z)=3x+2y+z+5 \qquad g(x,y,z)=9x^2+4y^2-z$

$\nabla f = \langle 3, 2, 1 \rangle$

$\nabla g = \langle 18x, 8y, -1 \rangle$

$\nabla f = \lambda \nabla g$

$\langle 3,2,1\rangle = \lambda \langle 18x, 8y, -1\rangle$

$3 = \lambda\, 18x$

$2 = \lambda\, 8y \qquad \lambda=-1$

$1 = -\lambda$

$$3 = -18x \Rightarrow x = -1/6$$

$$2 = -8y \Rightarrow y = -1/4$$

نعوض في معادلة القيد لحساب قيمة z

$$9x^2 + 4y^2 - z = 0 \Rightarrow 9\left(-\frac{1}{6}\right)^2 + 4\left(-\frac{1}{4}\right)^2 = z$$

$$z = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

النقطة الحرجة

$$\left(-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2}\right)$$

minimum

في حالة وجود اكثر من قيد

$$\nabla f = \lambda \nabla g$$

$$\nabla f = \mu \nabla h$$

$$\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$$

EXAMPLE 4 Find the maximum and minimum values of $f(x, y, z) = x + 2y + 3z$ on the ellipse that is the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $y + z = 1$ (see Figure 5).

$$\nabla f = \lambda \nabla g + \mu \nabla f$$

$f = x + 2y + 3z \qquad \nabla f = \langle 1, 2, 3 \rangle$

$g = x^2 + y^2 - 2 \qquad \nabla g = \langle 2x, 2y, 0 \rangle$

$h = y + z - 1 \qquad \nabla h = \langle 0, 1, 1 \rangle$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$\langle 1, 2, 3 \rangle = \lambda \langle 2x, 2y, 0 \rangle + \mu \langle 0, 1, 1 \rangle$$

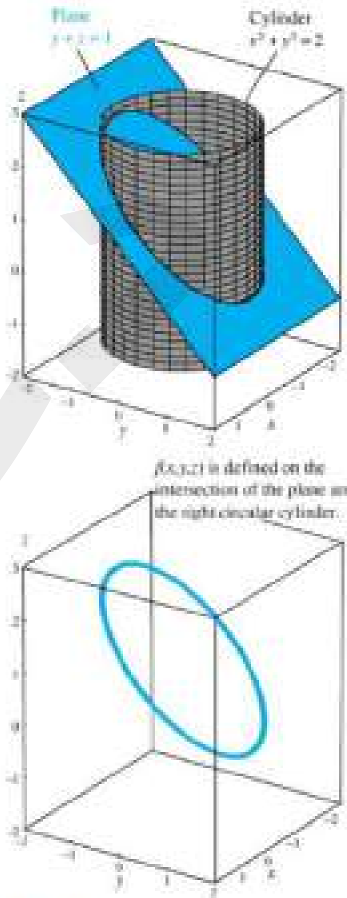


Figure 5

$1 = 2\lambda x \quad \text{--①}$

$2 = \lambda 2y + \mu \quad \text{--②}$

$3 = \mu \quad \text{--③}$

$x^2 + y^2 - 2 = 0$

$y + z - 1 = 0$

نعوض μ في معادلة 2

$2 = \lambda 2y + \mu$

$-1 = \lambda 2y$

الا ن معادله ١ و 2
اصبحت

$1 = 2\lambda x$

$-1 = 2\lambda y$

$-1 = \dfrac{x}{y}$

$$\boxed{x = -y}$$

نعوض في معادلة القيد

$$x^2 + y^2 - 2 = 0$$

$$x^2 + (-x)^2 = 2$$

$$2x^2 = 2 \qquad x^2 = 1$$

$$\boxed{x = \pm 1}$$

$$1^2 + y^2 = 2$$

$$y^2 = 1$$

$$y \begin{cases} +1 \\ -1 \end{cases}$$

(x, y, z)

$(1, -1, 2)$

$(-1, +1, 0)$

نعوض بالقيد لنأتي لحساب قيمة z

$$y + z - 1 = 0$$

$$-1 + z - 1 = 0 \qquad z = 2$$

$$+1 + z - 1 = 0 \qquad z = 0$$

النقاط الحرجة $(1, -1, 2)$ $(-1, 1, 0)$

نعوض النقاط في معادلة f للتأكد من القيم العظمى و الصغرى

$$f = x + 2y + 3z$$

	f	
$(1, -1, 2)$	5	$\Rightarrow$ Max
$(-1, 1, 0)$	1	$\Rightarrow$ min

Problem Set 12.9

1. Find the minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = xy - 3 = 0$.

$xy = 3$

$f(x,y) = x^2 + y^2 \qquad g(x,y) = xy - 3$

$\nabla f = \langle 2x, 2y \rangle \qquad \nabla g = \langle y, x \rangle$

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, 2y \rangle = \lambda \langle y, x \rangle$$

$$2x = \lambda y \qquad \frac{x}{y} = \frac{y}{x}$$

$$2y = \lambda x \qquad \boxed{x^2 = y^2}$$

$x = \pm y$

نعوض في معادلة القيد

$xy = 3$

$x^2 = 3$

$x = \pm\sqrt{3}$

$y = \pm\sqrt{3}$

$(\sqrt{3}, \sqrt{3}), (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}), (-\sqrt{3}, -\sqrt{3})$ النقاط الحرجة

$(\sqrt{3})^2 + (\sqrt{3})^2 = 6$ كل النقاط تعطي نتيجه

minimum

2. Find the maximum of $f(x, y) = xy$ subject to the constraint $g(x, y) = 4x^2 + 9y^2 - 36 = 0$.

$f(x,y) = xy$ $\qquad$ $g(x,y) = 4x^2 + 9y^2 - 36$

$\nabla f = \langle y, x \rangle$ $\qquad$ $\nabla g = \langle 8x, 18y \rangle$

$$\nabla f = \lambda \nabla g$$

$$\langle y, x \rangle = \lambda \langle 8x, 18y \rangle$$

$y = 8\lambda x$

$x = 18\lambda y$ $\qquad$ $\frac{y}{x} = \frac{4}{9}\frac{x}{y}$

$$\boxed{9y^2 = 4x^2}$$

نعوض في معادلة القيد

$4x^2 + 9y^2 = 36$

$9y^2 + 9y^2 = 36$ $\qquad$ $18y^2 = 36$ $\qquad$ $y^2 = 2$

$$\boxed{y = \pm\sqrt{2}}$$

$9(2) = 4x^2$ $\qquad$ $x^2 = \frac{18}{4} = \frac{9}{2} = \pm\frac{3}{\sqrt{2}}$

Critical points

$\left(+\frac{3}{\sqrt{2}}, +\frac{2}{\sqrt{2}}\right)$	3	$\Leftarrow$ max
$\left(-\frac{3}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right)$	3	$\Leftarrow$ max
$\left(-\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$	-3	$\Leftarrow$ min
$\left(\frac{3}{\sqrt{2}}, -\frac{2}{\sqrt{2}}\right)$	-3	$\Leftarrow$ min

3. Find the maximum of $f(x, y) = 4x^2 - 4xy + y^2$ subject to the constraint $x^2 + y^2 = 1$.

$f(x,y) = 4x^2 - 4xy + y^2$ $\qquad$ $g(x,y) = x^2+y^2-1$

$\nabla f = \langle 8x-4y \,,\, -4x+2y \rangle$ $\qquad$ $\nabla g = \langle 2x \,,\, 2y \rangle$

$$\nabla f = \lambda \nabla g$$

$$\langle 8x-2y \,,\, -4x+2y \rangle = \lambda \langle 2x \,,\, 2y \rangle$$

$8x - 4y = \lambda 2x$

$\div 2$ $\quad \cancel{4x} - \cancel{2y} = \lambda x$

$-\cancel{4x} + \cancel{2y} = 2\lambda y$

$\lambda x + 2\lambda y = 0$

$\lambda(x+2y) = 0$

$\lambda = 0$ $\qquad$ $x+2y=0$

$x = -2y$

$\lambda = 0 \implies 8x - 4y = 0$ $\qquad$ $\boxed{2x = y}$

نعوض في معادله القيد

$x^2 + (2x)^2 = 1$ $\qquad$ $5x^2 = 1$ $\qquad$ $x = \pm\frac{1}{\sqrt{5}}$

$y = 2x = \pm\frac{2}{\sqrt{5}}$

$$\boxed{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \,,\, \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)}$$

$x = -2y$ $\qquad$ $x^2 + y^2 = 1$ $\qquad$ $4y^2 + y = 1$

$5y^2 = 1$ $\qquad$ $y = \pm\frac{1}{\sqrt{5}}$

$y = \pm \frac{1}{\sqrt{5}}$ $\quad$ $x = \mp \frac{2}{\sqrt{5}}$

$$\boxed{\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) \quad \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)}$$

	$f = 4x^2 - 4xy + y^2$	
$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$	$4\left(\frac{1}{5}\right) - 4\left(\frac{2}{5}\right) + \frac{4}{5} = 0$	$\Rightarrow$ min
$\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$	0	$\Rightarrow$ min
$\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$	$4\left(\frac{4}{5}\right) + 4\left(\frac{2}{5}\right) + \frac{1}{5} = 5$	$\Rightarrow$ max
$\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$	5	$\Rightarrow$ max

4. Find the minimum of $f(x, y) = x^2 + 4xy + y^2$ subject to the constraint $x - y - 6 = 0$. $g = x - y - 6$

$\nabla f = \langle 2x + 4y, 4x + 2y \rangle$ $\quad$ $\nabla g = \langle 1, -1 \rangle$

$\nabla f = \lambda \nabla g$

$2x + 4y = \lambda$ $\quad$ $6x + 6y = 0$

$4x + 2y = -\lambda$ $\quad$ $\boxed{x = -y}$

$-y - y - 6 = 0$ $\quad$ نعوض في معادلة القيد

$-2y = 6$ $\quad$ $y = -3$ $\quad$ $x = 3$

critical $(3, -3)$ $\Rightarrow$ min

5. Find the minimum of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + 3y - 2z = 12$.

$x + 3y - 2z - 12 = 0$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla g = \langle 1, 3, -2 \rangle$$

$$\nabla f = \lambda \nabla g$$

$2x = \lambda \quad \ldots (1)$

$2y = 3\lambda \quad \ldots (2)$

$2z = -2\lambda \quad \ldots (3)$

$(1)/(2)$ $\quad \dfrac{2x}{2y} = \dfrac{1}{3}$ $\quad \boxed{y = 3x}$

$(1)/(3)$ $\quad \dfrac{2x}{2z} = \dfrac{1}{-2}$ $\quad \boxed{z = -2x}$

نعوض في معادلة القيد

$x + 3y - 2z = 12$

$x + 9x + 4x = 12$

$x = \dfrac{12}{14} = \dfrac{6}{7}$ $\qquad y = \dfrac{18}{7}$ $\qquad z = \dfrac{-12}{7}$

$$\left(\frac{6}{7}, \frac{18}{7}, \frac{12}{7} \right)$$

6. Find the minimum of $f(x, y, z) = 4x - 2y + 3z$ subject to the constraint $2x^2 + y^2 - 3z = 0$.

$\nabla f = \langle 4, -2, 3 \rangle$

$\nabla g = \langle 4x, 2y, -3 \rangle$

$4 = 4x\lambda$ ---- ①

$-2 = 2y\lambda$ ---- ②

$3 = -3\lambda$ ---- ③

③ $\lambda = -1$

① $x = -1$

② $y = 1$

$(-1, 1, z)$

نعوض في معادله القيد

$2x^2 + y^2 - 3z = 0$

$2 + 1 - 3z = 0$

$z = 1$

Critical point $(-1, 1, 1)$

$f(-1, 1, 1) = -3$

min