Differential Equations and Linear Algebra(MTH2143)

CHAPTER1- NOTES

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1 First-Order Differential Equations

معادلات تفاضليه

1.1 Differential Equations and Mathematical Models

$$x^2 + x = 6 \qquad x = 2$$

$$4 + 2 = 6$$

$$\frac{dy}{dx} + x = 6$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6x$$

اي معادله تحتوي على مشتقات تسمى معادله تفاضليه

$$\frac{dx}{dt} + 3x = 7t$$

$x^2 + x = 6$ $\boxed{x = 2}$ slowtion

تمو المعادلات التفاضليه عادةً الحل يكون عائله لانهائيه من حلول مختلفة

$$y(x) = c e^x$$

c ثابتة

Example 1 The differential equation

$x(t)$ = المجهول هو

مشتقة الاولى

$$\frac{dx}{dt} = x^2 + t^2$$

x' $\quad \frac{dx}{dt}$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$y(x)$ المجهول

مشتقة اولى ومشتقة ثانية

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives y' and y'' of y.

Proof that $y(x)$ is a solution of the DE

Example 2 If C is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

$$y(x) = ce^{x^2} \qquad \frac{dy}{dx} = 2xy$$

للتأكد ان y تحقق المعادلة يجب تطبيق y في المعادلة

$$y(x) = ce^{x^2}$$

infinite family of different solution

$$\frac{dy}{dx} = c(2xe^{x^2})$$

$$= 2x\,\cancel{ce^{x^2}}$$

$$\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx} = 2xy$$

$$2xy = 2xy$$

f	f'
e^x	e^x
e^{3x}	$3e^{3x}$
e^{x^2}	$2xe^{x^2}$
e^{4x^2}	$8xe^{4x^2}$

Example 5 **Population growth** The *time rate of change* of a population $P(t)$ with constant birth and death rates is, in many simple cases, proportional to the size of the population. That is,

$$\frac{dP}{dt} = kP, \tag{6}$$

where k is the constant of proportionality. ■

Let us discuss Example 5 further. Note first that each function of the form

$$P(t) = Ce^{kt} \tag{7}$$

$$\frac{dP}{dt} = kP \qquad P(t) = Ce^{kt}$$

عدد كبير من الحلول

هل نعتبر $P(t)$ حلا للمعادلة التفاضلية

$$\frac{dP}{dt} = Cke^{kt} = k\underbrace{Ce^{kt}}_{} = kP$$

$$\frac{dP}{dt} = kP$$

$$kP = kP$$

$P(t)$ نعتبر حلا صحيحا للمعادلة:

Example 7 If C is a constant and $y(x) = 1/(C - x)$, then

$$\frac{dy}{dx} = \frac{1}{(C - x)^2} = y^2$$

if $x \neq C$. Thus

$$y(x) = \frac{1}{C - x} \tag{8}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \tag{9}$$

$$y(x) = \frac{1}{C-x} \qquad \frac{dy}{dx} = y^2$$

$$\frac{dy}{dx} = \frac{+1}{(C-x)^2} =$$

$$= \frac{1}{C-x} \cdot \frac{1}{C-x} = y^2$$

f	f'
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{2+3x}$	$-\frac{3}{(2+3x)^2}$

$y(x) = \frac{1}{C-x}$ is a valid solution

قيمه C محدده وتساوي 1

$$y(x) = \frac{1}{1-x}$$

في حال اعطائي السؤال شرط حول y نعوض
هذا الشرط لحساب قيمة C

Example 9 If A and B are constants and

$$\boxed{y(x) = A\cos 3x + B\sin 3x,} \tag{14}$$

then two successive differentiations yield

$$y'(x) = -3A\sin 3x + 3B\cos 3x,$$
$$y''(x) = -9A\cos 3x - 9B\sin 3x = -9y(x)$$

for all x. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$\boxed{y'' + 9y = 0} \tag{15}$$

on the whole real number line. Figure 1.1.6 shows the graphs of several such solutions. ■

لنتأكد ان y يعتبر حلاً للمعادله يجب تطبيقه فيه

$$y'' + 9y = 0$$

$$y = A\cos 3x + B\sin 3x$$

$$y' = -3A\sin 3x + B\,3\cos 3x$$

$$y'' = -9A\cos 3x - 9B\sin 3x$$

$$-9A\cos 3x - 9B\sin 3x + 9(A\cos 3x + B\sin 3x)$$

$$-9A\cancel{\cos 3x} - 9B\cancel{\sin 3x} + 9A\cancel{\cos 3x} + 9B\cancel{\sin 3x}$$

$$= \text{Zero}$$

$$y'' + 9y = 0$$
$$0 = 0 \;\checkmark$$

y هو حل للمعادلة

Example 10 Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(\overset{x=1}{1}) = 2. \rightarrow y=2$$

$$y(x) = \frac{1}{c-x} \qquad \frac{dy}{dx} = y^2$$

نعوض الشرط في الحل

$$\underset{2}{y}(x) = \frac{1}{c-\underset{1}{x}}$$

$$2 = \frac{1}{c-1}$$

$$2(c-1) = 1$$

$$2c-2 = 1$$

$$2c = 3$$

$$\boxed{c = \frac{3}{2}}$$

$$\boxed{y = \frac{1}{\frac{3}{2} - x}}$$

1.1 Problems

In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.

1. $y' = 3x^2$; $y = x^3 + 7$

$y = x^3 + 7$ $\qquad$ $y' = 3x^2$

$y' = 3x^2$

$3x^2 = 3x^2$

$\therefore$ y is a solution

2. $\boxed{y' + 2y = 0}$; $y = 3e^{-2x}$

$y = 3e^{-2x}$ $\qquad$ $y' = 3(-2)e^{-2x} = -6e^{-2x}$

$y' + 2y = 0$

$-6e^{-2x} + 2(3)e^{-2x} \stackrel{?}{=} 0$

$-6e^{-2x} + 6e^{-2x} = 0$

$0 = 0$

6. $y'' + 4y' + 4y = 0;\ y_1 = e^{-2x},\ y_2 = xe^{-2x}$

$y_1 = e^{-2x}$ نبدأ بالحل الاول

$y_1' = -2e^{-2x}$ $\quad$ $y_1'' = 4e^{-2x}$

$$y'' + 4y' + 4y = 0$$

$$4e^{-2x} + 4(-2e^{-2x}) + 4e^{-2x} \overset{?}{=} 0$$

$$4e^{-2x} - 8e^{-2x} + 4e^{-2x} \overset{?}{=} 0$$

$$e^{-2x}\cancel{(4-8+4)} = 0$$

$0 = 0$ $\quad$ $y_1 \Rightarrow$ Solution

$y_2 = xe^{-2x}$ ننطلق الحل الثاني

$y_2' = -2xe^{-2x} + e^{-2x}$

$y_2'' = 4xe^{-2x} - 2e^{-2x} - 2e^{-2x}$

$y_2'' = 4xe^{-2x} - 4e^{-2x}$

$$y'' + 4y' + 4y = 0$$

$$\cancel{4xe^{-2x}} - \cancel{4e^{-2x}} - \cancel{8xe^{-2x}} + \cancel{4e^{-2x}} + \cancel{4xe^{-2x}}$$

$0 = 0$ $\quad$ Yes its solation

7. $y'' + 2y' + 5y = 0;\ y_1 = e^{-x}\cos 2x,\ y_2 = e^{-x}\sin 2x$

(a) (b)

(a)

$y_1 = e^{-x}\cos 2x \qquad y_1' = -2e^{-x}\sin 2x - e^{-x}\cos 2x$

$y_1'' = -4e^{-x}\cos 2x + 2e^{-x}\sin 2x + 2e^{-x}\sin 2x$

$+ e^{-x}\cos 2x$

$y'' = -3e^{-x}\cos 2x + 4e^{-x}\sin 2x$

نعوض في المعادلة

$y'' + 2y' + 5y = 0$

$e^{x}\left[-3\cancel{\cos 2x} + 4\cancel{\sin 2x} - 4\cancel{\sin 2x} - 2\cancel{\cos 2x} + 5\cancel{\cos 2x}\right]$

$0 = 0 \qquad y_1 = e^{-x}\cos 2x$ is solution ✓

$y_2 = e^{-x}\sin 2x \qquad y_2' = 2e^{-x}\cos 2x - e^{-x}\sin 2x$

$y_2'' = -4e^{-x}\sin 2x - 2e^{-x}\cos 2x - 2e^{-x}\cos 2x$

$+ e^{-x}\sin 2x$

$y_2'' = -3e^{-x}\sin 2x - 4e^{-x}\cos 2x$

$y'' + 2y' + 5y = 0$ نعوض في المعادلة

$$e^{-x}\left[-3\cancel{\sin 2x}-4\cancel{\cos 2x}+4\cancel{\cos 2x}-2\sin 2x\right.$$
$$\left.+5\cancel{\sin 2x}\right]=0$$
$$0=0 \quad \checkmark$$

In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.

13. $3y' = 2y$

$$y=e^{rx} \qquad y'=re^{rx}$$

$$3y'=2y$$

$$3r\cancel{e^{rx}}=2\cancel{e^{rx}}$$

$$3r=2$$

$$\boxed{r=\frac{2}{3}}$$

$$y=e^{\frac{2}{3}x}$$

In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.

17. $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$

① نتأكد أن $y(x)$ يحقق المعادلة:

$$y(x) = Ce^{-x}$$

$$y'(x) = -Ce^{-x}$$

$$y' + y = 0$$

$$\cancel{-Ce^{-x}} + \cancel{Ce^{-x}} = 0$$

$$\boxed{0 = 0}$$

y satisfies the DE

② نعوض الشرط الابتدائي لحساب قيمة C

$$y(0) = 2$$

$x = 0$ $\quad$ $y = 2$

$$y = Ce^{-x}$$

$$2 = C\underbrace{\cancel{e^{0}}}_{1}$$

$$\boxed{C = 2}$$

$$\boxed{y = 2e^{-x}}$$

20. $\underbrace{y' = x - y}_{\text{معادله}};\ \underbrace{y(x) = Ce^{-x} + x - 1}_{\text{حل}},\ \underbrace{y(0) = 10}_{\text{شرط}}$

① نتأكد ان y يحقق المعادلة:

$$y = ce^{-x} + x - 1 \qquad y' = -ce^{-x} + 1$$

$$y' = x - y$$

$$-ce^{-x} + 1 = \cancel{x} - ce^{-x} - \cancel{x} + 1$$

$$-ce^{-x} + 1 = -ce^{-x} + 1$$

y satisfies the DE

② نستخدم الشرط لحساب C

$$x = 0 \qquad y = 10$$

$$y = ce^{-x} + x - 1$$

$$10 = c\cancel{e^{0}} + 0 - 1$$

$$10 = c - 1 \qquad \boxed{C = 11}$$

$$\boxed{y = 11e^{-x} + x - 1}$$

35. In a city having a fixed population of P persons, the time rate of change of the number N of those persons who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor.

$$\frac{dN}{dt} =$$ معدل التغير في عدد الاشخاص الذين سمعو الاشاعه

= يتناسب طردي مع عدد الاشخاص الذين لم سمعو الاشاعه

$$K(P-N)$$

$$\frac{dN}{dt} = K(P-N)$$

In Problems 37 through 42, determine by inspection at least one solution of the given differential equation. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

38. $y' = y$

هناك y عندما اشتقها تبقى نفسها

$$\boxed{y = e^x}$$

42. $y'' + y = 0$ $\Rightarrow$ $y'' = -y$

هناك داله y عندما نشتقها مرتين و نجمعها مع نفسها تكون الجواب صفر

$$\boxed{y = \cos x} \qquad y' = -\sin x \qquad \underline{y'' = -\cos x}$$

$$y'' + y = 0$$

$$-\cos x + \cos x = 0$$

1.2 Integrals as General and Particular Solutions

$$\frac{dy}{dx} = f(x)$$

المعادله التفاضليه بهذا الشكل تحل باستخدام التكامل

$$\int dy = \int f(x)\, dx$$

$$y(x) = G(x) + C$$

مثال

$$\frac{dy}{dx} = 5x$$

$$\int dy = \int 5x\, dx$$

$$\boxed{y = \frac{5}{2}x^2 + C}$$

$\int x^3 = \frac{x^4}{4}$	$\int 5dx = 5x$
$\int 3x^5 = \frac{3x^6}{6}$	$\int 10dx = 10x$

Example 1 Solve the initial value problem

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2.$$

$x = 1$

$y = 2$

$$\frac{dy}{dx} = 2x + 3$$

$$\int dy = \int (2x + 3)\, dx$$

$$\boxed{y = x^2 + 3x + C}$$

الان نعوض الشرط حتى نحسب C

$$y = x^2 + 3x + C$$
$$2 = 1^2 + 3(1) + C$$
$$2 = 4 + C$$

$$C = 2 - 4 = -2$$

$$\boxed{C = -2}$$

$$\boxed{y = x^2 + 3x - 2}$$

Velocity and acceleration

$$v = \frac{dx}{dt}$$

مثال

$$V = 2t^2 \Longrightarrow \frac{dx}{dt} = 2t^2$$

$$\int dx = \int 2t^2 \, dt$$

$$\boxed{x = \frac{2t^3}{3} + c}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

مثال 2.

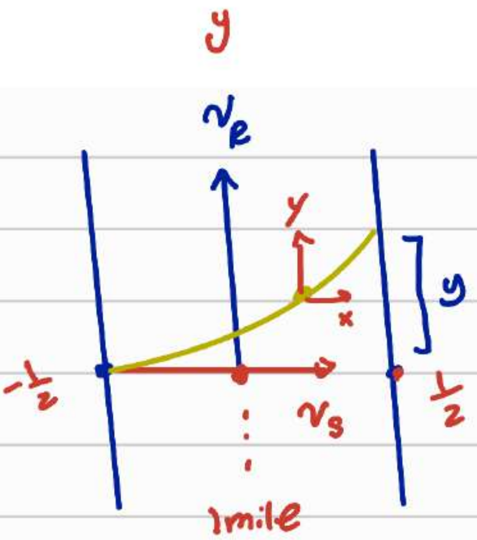
$$a = 7t \qquad \frac{dv}{dt} = 7t$$

$$\int dv = \int 7t \, dt$$

$$\boxed{V = \frac{7t^2}{2} + c}$$

Example 4 **River crossing** Suppose that the river is 1 mile wide and that its midstream velocity is $v_0 = 9$ mi/h. If the swimmer's velocity is $v_S = 3$ mi/h, then Eq. (19) takes the form

$$\frac{dy}{dx} = 3(1 - 4x^2).$$



$\frac{dy}{dx}$ = المعادله التفاضليه لحركة السباح

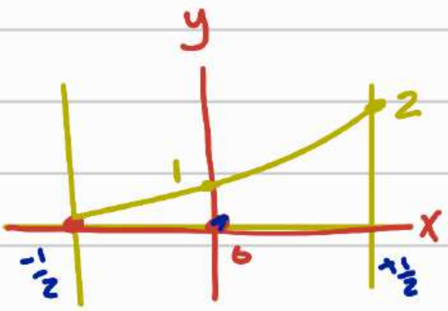
$$\frac{dy}{dx} = 3(1-4x^2)$$

ثم سينجرف السباح عند وصوله الى الطرف الثاني

(المطلوب حساب y)

$$\int dy = \int 3 - 12x^2 \, dx$$

$$\boxed{y = 3x - 4x^3 + C}$$



$x = -\frac{1}{2}$ نقطة البداية

$y = 0$

نعوض الشرط لحساب C

$x = -\frac{1}{2}$ $y = 0$

$$y = 3x - 4x^3 + C$$

$$0 = 3\left(-\tfrac{1}{2}\right) - 4\left(-\tfrac{1}{2}\right)^3 + C$$

$$0 = -\frac{3}{2} + \frac{4}{8} + C$$

$$0 = -\frac{3}{2} + \frac{1}{2} + C$$

$$0 = -1 + C \qquad \boxed{C = 1}$$

$$\boxed{y = 3x - 4x^3 + 1}$$

معادلة موقع للسباح عند أي نقطة

$y = 1 \leftarrow x = 0$ في منتصف النهر

$x = \frac{1}{2}$ على طرف النهر (النهاية)

$$y = 3\left(\tfrac{1}{2}\right) - 4\left(\tfrac{1}{2}\right)^3 + 1$$

$$y = \frac{3}{2} - \frac{1}{2} + 1 = 2$$

1.2 Problems

In Problems 1 through 10, find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.

1. $\frac{dy}{dx} = 2x + 1;\ y(0) = 3$

$x = 0$

$y = 3$

$$\frac{dy}{dx} = 2x + 1$$

$$\int dy = \int (2x+1)\, dx$$

$$y = \cancel{2}\frac{x^2}{\cancel{2}} + x + C$$

$$y = x^2 + x + C$$

نعوض الشرط لحساب C

$$y = x^2 + x + C$$

$$3 = 0 + 0 + C$$

$$\boxed{3 = C}$$

$$\boxed{y = x^2 + x^2 + 3}$$

3. $\frac{dy}{dx} = \sqrt{x};\ y(4) = 0$ — $x = 4$, $y = 0$

$$\frac{dy}{dx} = x^{1/2}$$

$$\int dy = \int x^{1/2}\, dx$$

$$y = \frac{2}{3}x^{3/2} + C$$

نعوض الشرط لحساب C

$$0 = \frac{2}{3}(4)^{3/2} + C$$

$$0 = \frac{2}{3}(8) + C \qquad 0 = \frac{2}{3}(8) + C$$

$$C = -\frac{2}{3}(8)$$

$$y = \frac{2}{3}x^{3/2} - \frac{2}{3}(8)$$

$$\boxed{y = \frac{2}{3}\left(x^{3/2} - 8\right)}$$

5. $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}$; $y(2) = -1$ $\begin{cases} x=2 \\ y=-1 \end{cases}$

$\int x\,dx = \frac{x^2}{2} + c$

$\int x^2\,dx = \frac{x^3}{3} + c$

$\int (2x+1)^2\,dx = \frac{(2x+1)^3}{6}$

$\int (3x+5)^{1/2}\,dx = \frac{2}{9}(3x+5)^{3/2}$

$$\int dy = \int (x+2)^{-1/2}\,dx$$

$$\underbrace{y = 2(x+2)^{1/2} + C}$$

نعوض الشرط لحساب قيمة C

$$y = 2(x+2)^{1/2} + C$$

$$-1 = 2(2+2)^{1/2} + C$$

$$-1 = 2\sqrt{4} + C$$

$$-1 = 4 + C$$

$$C = -1-4 = -5$$

$$\boxed{C = -5}$$

$$\boxed{y = 2\sqrt{x+2} - 5}$$

10. $\frac{dy}{dx} = xe^{-x};\ y(0) = 1$ $\rightarrow x = 0$, $\rightarrow y = 1$

$\int xe^{x}$ $\int x\sin x$ $\int x\cos x$ $\int x\ln x$

كلها تحل باستخدام بالأجزاء

$$\frac{dy}{dx} = xe^{-x}$$

$$\int dy = \int xe^{-x}\,dx$$

$\int u\,dv = uv - \int v\,du$

$$y = \int \underbrace{x}_{u}\,\underbrace{e^{-x}}_{dv}\,dx$$

$u = x$ $dv = e^{-x}$

$du = dx$ $v = -e^{-x}$

$$\int u\,dv = uv - \int v\,du$$
$$= -xe^{-x} - \int -e^{-x}\,dx$$
$$= -xe^{-x} + \int e^{-x}\,dx$$
$$= -xe^{-x} - e^{-x}$$
$$= e^{-x}(-x-1)$$

$$\boxed{y = e^{-x}(-x-1) + c}$$

نعوض الشرط لحساب C

$$y = e^{-x}(-x-1) + C$$
$$1 = \cancel{e^{-0}}(-0-1) + c$$
$$1 = -1 + C$$
$$C = 1 + 1 = 2$$

$$\boxed{y = e^{-x}(-x-1) + 2}$$

In Problems 11 through 18, find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and initial velocity $v_0 = v(0)$.

11. $a(t) = 50$, $v_0 = 10$, $x_0 = 20$

$a = \frac{dv}{dt}$ $\qquad$ $V = \frac{dx}{dt}$

$\frac{dv}{dt} = 50 \Rightarrow \int dv = \int 50\, dt$

$\boxed{V = 50t + C_1}$

نعوض الشرط الخاص بالسرعة
لحساب C_1

لحظة البداية كانت السرعة 10
$V = 10$ $\qquad$ عند $t = 0$

$V = 50t + C_1$

$10 = 50(0) + C_1$

$\boxed{C_1 = 10}$

معادلة السرعة

$\boxed{V = 50t + 10}$

حساب X

$\frac{dx}{dt} = 50t + 10$

$\int dx = \int (50t + 10)\, dt$

$$X = \frac{50}{2}t^2 + 10t + c_2$$

نعوض الشرط الخاص ب x لحساب c_2

الموقع لحظة البداية $= 20$

$$t = 0 \qquad X = 20$$

$$x = 25t^2 + 10t + c_2$$

$$20 = 0 + 0 + c_2$$

$$c_2 = 20$$

$$\boxed{X(t) = 25t^2 + 10t + 20}$$

$$\begin{tikzcd}
\text{موقع } X(t) \arrow[r, bend left, "\text{مشتقة}"] & \text{السرعة } v(t) \arrow[r, bend left, "\text{مشتقة}"] \arrow[l, bend left, "\text{تكامل}"] & \text{التسارع } a(t) \arrow[l, bend left, "\text{تكامل}"]
\end{tikzcd}$$

15. $a(t) = 4(t+3)^2,\ v_0 = -1,\ x_0 = 1$

$$a = \frac{dv}{dt} = 4(t+3)^2$$

$$\int dv = \int 4(t+3)^2\,dt$$

$$v = \frac{4}{3}(t+3)^3 + c_1$$

$v = -1$ $\quad t = 0$ $\quad$ نعوض الشرط الخاص بالسرعة

$$-1 = \frac{4}{3}(0+3)^3 + c_1$$

$$-1 = 36 + c_1 \qquad \boxed{c_1 = -37}$$

$$\boxed{v = \frac{4}{3}(t+3)^3 - 37}$$

$$v = \frac{dx}{dt} = \frac{4}{3}(t+3)^3 - 37$$

$$\int dx = \int \frac{4}{3}(t+3)^3 - 37$$

$$X = \frac{\cancel{4}}{3}\,\frac{(t+3)^4}{\cancel{4}} - 37t + C_2$$

$$X = \frac{1}{3}(t+3)^4 - 37t + C_2$$

نعوض شرط الوضع الابتدائي $t=0$ $x=1$

$$1 = \frac{1}{3}(3)^4 - 37(0) + C_2$$

$$1 = 27 + C_2$$

$$\boxed{C_2 = -26}$$

$$\boxed{X = \frac{1}{3}(t+3)^4 - 37t - 26}$$

1.4 Separable Equations and Applications

الدرس الماضي

$$\frac{dy}{dx} = f(x) \qquad \int dy = \int f(x)\, dx$$

مثال

$$\frac{dy}{dx} = -6x \qquad \int dy = \int -6x \, dx$$

$$\boxed{y = -3x^2 + c}$$

طريقه فصل المتغيرات

$$\frac{dy}{dx} = f(x, y)$$

خطوات حل المعادله (طريقه الفصل)

① فصل جميع قيم x في طرف و y في طرف اخر

② تكامل الطرفين ويحاول كتابه المعادله بدلاله y

③ حساب قيمه الثابت C اذا توفرت الشروط

مثال

Solve the DE if $y(0) = 7$

$$\frac{dy}{dx} = -6xy$$

$$\int \frac{dy}{y} = \int -6x \, dx$$

$$\int \frac{1}{y}\,dy = -6\int x\,dx$$

$$\int \frac{1}{x}\,dx = \ln x$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x)$$

$$\ln y = -3x^2 + C$$

$$\cancel{e}^{\cancel{\ln} y} = e^{-3x^2+C}$$

$$y = e^{-3x^2} \cdot \underbrace{\cancel{e}^{C}}_{A}$$

$$y = Ae^{-3x^2}$$

نطبق الشرط لحساب الثابت

$x=0$ $y=7$

$$7 = A\cancel{e}^{\cancel{0}} \qquad A = 7$$

$$\boxed{y = 7e^{-3x^2}}$$

Example 1 Solve the differential equation

$$\frac{dy}{dx}=\frac{4-2x}{3y^2-5}, \quad y(1)=3$$

نستخدم طريقة فصل المتغيرات لحل المعادلة:

$$\frac{dy}{dx}=\frac{4-2x}{3y^2-5}$$

$$\int(3y^2-5)\,dy=\int(4-2x)\,dx$$

$$y^3-5y=4x-x^2+C$$

$x=1 \quad y=3$ نطبق الشرط

$$3^3-5(3)=4(1)-1^2+C$$

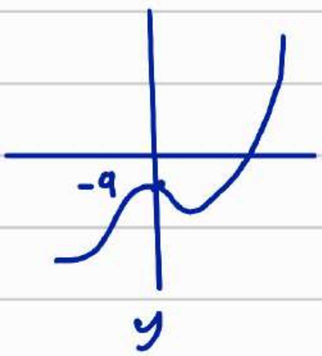
$$27-15-4+1=C$$

$$C=9$$

$$y^3-5y=4x-x^2+9$$

$$\boxed{y^3-5y-4x+x^2=9}$$

مثال اذا افترضنا $x=0$



$$y^3-5y-9=0$$

Example 2 Find all solutions of the differential equation

$$\frac{dy}{dx} = 6x(y-1)^{2/3}.$$

$y(1) = 1$

$$\frac{dy}{dx} = 6x\,(y-1)^{2/3}$$

$$\int \frac{dy}{(y-1)^{2/3}} = \int 6x\,dx$$

$$\int (y-1)^{-2/3}\,dy = \int 6x\,dx$$

$$\cancel{3}(y-1)^{\frac{1}{3}} = \cancel{3}x^2 + c$$

$$\left((y-1)^{1/\cancel{3}}\right)^{\cancel{3}} = \left(x^2 + c\right)^3$$

$$y - 1 = (x^2 + c)^3$$

$$\boxed{y = (x^2 + c)^3 + 1}$$

نطبق الشرط لحساب C

$\boxed{y(1) = 1}$ $\quad x=1 \quad y=1$

$$1 = (1^2 + C)^3 + 1$$

$$(1 + C) = 0$$

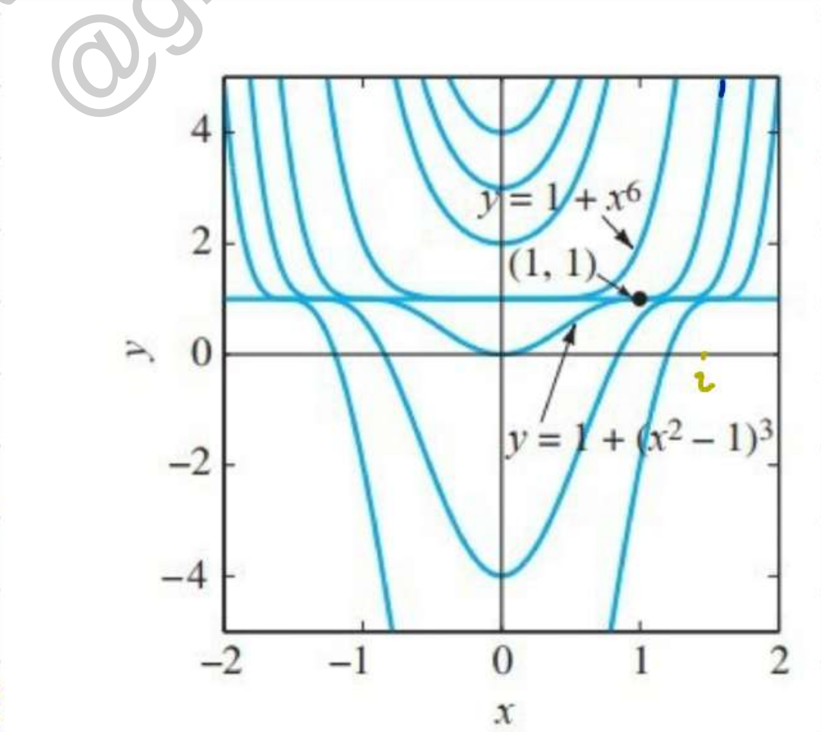
$$\boxed{C = -1}$$

$$\boxed{y = (x^2 - 1)^3 + 1}$$

$y(1)=1$
$C=-1$

the solution is not unique

there is a family of solution



عند حساب حل المعادله التفاضليه وكانت النتيجه مجموعه من الحلول
اذا الداله غير قابله للاشتقاق

the function $f(x,y)$ is not differentiable

1.4 Problems

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.

1. $\frac{dy}{dx} + 2xy = 0$

$$\frac{dy}{dx} = -2xy$$

$$\int \frac{dy}{y} = \int -2x\,dx$$

$$\ln y = -x^2 + C$$

$$e^{\cancel{\ln} y} = e^{-x^2+C}$$

$$y = e^{-x^2} e^{C}$$

$$y = Ae^{-x^2}$$

2. $\dfrac{dy}{dx} + 2xy^2 = 0$

$$\frac{dy}{dx} = -2xy^2$$

$$\int \frac{dy}{y^2} = \int -2x \; dx$$

$$\int \frac{1}{y^2} = -\frac{1}{y}$$

$$\int y^{-2} dy = -\int 2x \, dx$$

$$-y^{-1} = -\frac{\cancel{2}}{\cancel{2}} x^2 + C$$

$$\frac{1}{y} = x^2 + C$$

$$y = \frac{1}{x^2 + C}$$

3. $\frac{dy}{dx} = y \sin x$

$$\int \frac{dy}{y} = \int \sin x \, dx$$

$$\ln y = -\cos x + c$$

$$e^{\ln y} = e^{-\cos x + c}$$

$$y = e^{-\cos x} e^{c}$$

$$y = A e^{-\cos x}$$

4. $(1+x)\dfrac{dy}{dx} = 4y$

$$\frac{(1+x)}{1+x}\,\frac{dy}{4y} = \frac{dx}{1+x}$$

$$\int \frac{dy}{y} = \int 4\,\frac{dx}{1+x}$$

$$\ln y = \ln|1+x|^4 + c$$

$$\cancel{e}^{\cancel{\ln y}} = \cancel{e}^{\cancel{\ln}(1+x)^4}\, e^c$$

$$y = A(1+x)^4$$

$$\int \frac{1}{x}\,dx = \ln x$$

$$\int \frac{f'}{f(x)} = \ln f(x)$$

$$\int \frac{3}{3x+5} = \ln 3x+5$$

$$\int \frac{1}{1+x} = \ln|1+x|$$

5. $2\sqrt{x}\,\frac{dy}{dx} = \sqrt{1-y^2}$

$$\frac{\cancel{2\sqrt{x}}}{\cancel{2\sqrt{x}}}\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{2\sqrt{x}} \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{2\sqrt{x}}$$

$$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\sin^{-1}(y) = \frac{1}{2}\int x^{-1/2}\,dx$$

$$\sin^{-1} y = \frac{1}{\cancel{2}}\,\frac{x^{1/2}}{\cancel{\frac{1}{2}}} + C$$

$$\sin^{-1} y = \sqrt{x} + C$$

$$\boxed{y = \sin\left(\sqrt{x} + C\right)}$$

$$\sin^{-1} x = A$$

$$\sin A = x$$

6. $\frac{dy}{dx} = 3\sqrt{xy} \sim \sqrt{x}\sqrt{y}$

$$\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x}\, dx$$

$$\int y^{-1/2}\, dy = 3 \int x^{1/2}\, dx$$

$$2y^{1/2} = 2x^{3/2} + c$$

$$y^{1/2} = x^{3/2} + A$$

$$\left(\sqrt{y}\right)^2 = \left(x^{3/2} + A\right)^2$$

$$y = \left(x^{3/2} + A\right)^2$$

10. $(1+x)^2 \dfrac{dy}{dx} = (1+y)^2$

$$\int \frac{dy}{(1+y)^2} = \int \frac{dx}{(1+x)^2}$$

$$\int (1+y)^{-2}\, dy = \int (1+x)^{-2}\, dx$$

$$-(1+y)^{-1} = -(1+x)^{-1} + c$$

$$\frac{1}{1+y} = \frac{1}{1+x} - \frac{c(1+x)}{1+x}$$

$$\frac{1}{1+y} = \frac{1-c(1+x)}{1+x}$$

$$1+y = \frac{1+x}{1-c(1+x)}$$

$$y = \frac{1+x}{1-c(1+x)} - 1$$

$$y = \frac{1+x - (1-c(1+x))}{1-c(1+x)}$$

$$y = \frac{x+c(1+x)}{1-c(1+x)}$$

11. $y' = xy^3$

$$\frac{dy}{dx} = xy^3 \Rightarrow \int \frac{dy}{y^3} = \int x\,dx$$

$$\int y^{-3}\,dy = \int x\,dx$$

$$-\frac{y^{-2}}{2} = \frac{x^2}{2} + c$$

$$\frac{1}{y^2} = -x^2 - C$$

$$y^2 = \frac{1}{-x^2 - c}$$

$$y = \frac{1}{\sqrt{-x^2 - c^2}}$$

12. $yy' = x(y^2 + 1)$

$$y \frac{dy}{dx} = x(y^2+1)$$

$$\frac{1}{2}\int \frac{2y}{y^2+1}\, dy = \int x \, dx$$

$$\int \frac{f'}{f} = \ln f$$

$$\frac{1}{2}\ln|y^2+1| = \frac{x^2}{2} + \frac{c}{2}$$

$$\ln|y^2+1| = x^2 + c$$

$$\cancel{e}^{\cancel{\ln}(y^2+1)} = e^{x^2} e^{c}$$

$$y^2+1 = Ae^{x^2}$$

$$y^2 = Ae^{x^2} - 1$$

$$y = \sqrt{Ae^{x^2}-1}$$

$$\int \frac{y}{y^2+1}\, dy$$

$$u = y^2+1$$

$$\frac{du}{dy} = 2y$$

$$dy = \frac{du}{2y}$$

$$\int \frac{\cancel{y}}{u} \frac{du}{2\cancel{y}}$$

$$\frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln u$$

$$\frac{1}{2}\ln(y^2+1)$$

Find explicit particular solutions of the initial value problems in Problems 19 through 28.

19. $\dfrac{dy}{dx} = ye^x, \quad y(0) = 2e$

$$\frac{dy}{dx} = ye^x$$

$$\int \frac{dy}{y} = \int e^x \, dx$$

$$\ln y = e^x + c$$

$$\cancel{e^{\ln y}} = e^{e^x} e^c$$

$$y = Ce^{e^x}$$

$y(0) = 2e$ الان نطبق الشرط لحساب قيمة الثابت C

$$2\cancel{e} = C\cancel{e} \qquad C = 2$$

$$\boxed{y = 2e^{e^x}}$$

22. $\dfrac{dy}{dx} = 4x^3y - y, \quad y(1) = -3$

$$\frac{dy}{dx} = 4x^3y - y$$

$$\frac{dy}{dx} = y(4x^3 - 1)$$

$$\int \frac{dy}{y} = \int (4x^3 - 1)\,dx$$

$$\ln y = x^4 - x + C$$

$$\cancel{e}^{\cancel{\ln} y} = e^{x^4 - x}\, C$$

$$\boxed{y = C e^{x^4 - x}}$$

الان نطبق الشرط

$x = 1 \qquad y = -3$

$$-3 = C \cancel{e^{0}}^{\,1} \qquad C = -3$$

$$\boxed{y = -3e^{x^4 - x}}$$

Problems 29 through 32 explore the connections among general and singular solutions, existence, and uniqueness.

29. **(a)** Find a general solution of the differential equation $dy/dx = y^2$. **(b)** Find a singular solution that is not included in the general solution. **(c)** Inspect a sketch of typical solution curves to determine the points (a, b) for which the initial value problem $y' = y^2$, $y(a) = b$ has a unique solution.

$$\frac{dy}{dx} = y^2 \implies \int \frac{dy}{y^2} = \int dx$$

$$\int y^{-2}\,dy = \int dx$$

$$-y^{-1} = x + c$$

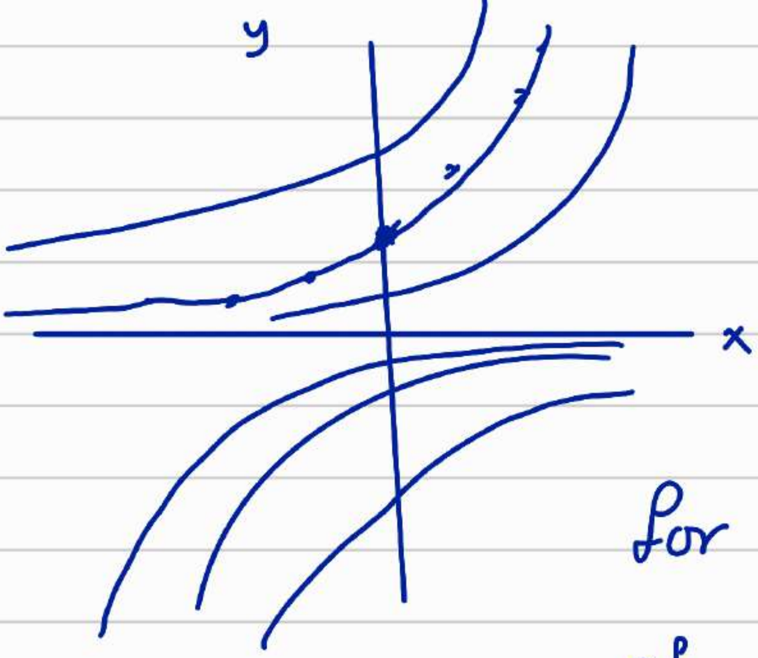
$$\frac{1}{y} = -(x + c) \qquad \boxed{y = \frac{-1}{x + C}}$$

هناك قيمة واحده لايمكن ان تكون حل للمعادله

$$y = 0$$

$$x = 0 \qquad y = -\frac{1}{0 + c} = -\frac{1}{c}$$

$x = 0$ $\quad c = -1,\ c = -2,\ c = -3 \quad$ $y = 1, \frac{1}{2}, \frac{1}{3}$



for each value
of x and y
there is a unique
sloution

1.5 Linear First-Order Equations

THEOREM 1 The Linear First-Order Equation

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0 \tag{11}$$

has a unique solution $y(x)$ on I, given by the formula in Eq. (6) with an appropriate value of C.

Remark 1 Theorem 1 gives a solution on the *entire* interval I for a *linear* differential equation, in contrast with Theorem 1 of Section 1.3, which guarantees only a solution on a possibly smaller interval.

Remark 2 Theorem 1 tells us that every solution of Eq. (3) is included in the general solution given in Eq. (6). Thus a *linear* first-order differential equation has *no* singular solutions.

Remark 3 The appropriate value of the constant C in Eq. (6)—as needed to solve the initial value problem in Eq. (11)—can be selected "automatically" by writing

$$\begin{aligned} \rho(x) &= \exp\left(\int_{x_0}^{x} P(t)\,dt\right), \\ y(x) &= \frac{1}{\rho(x)}\left[y_0 + \int_{x_0}^{x} \rho(t)Q(t)\,dt\right]. \end{aligned} \tag{12}$$

* طريقه حل المعادلات التفاضليه الخطيه من الدرجه الاولى

$$\frac{dy}{dx} + \underbrace{P(x)}_{\text{داله تحتوي فقط على } x \text{ او ثابت}}\, y = \underbrace{Q(x)}_{\text{داله تحتوي فقط على } x}$$

أي المعادلات التالية تعتبر خطية وإذا كانت خطية حدد $P(x)$ و $Q(x)$

① $\dfrac{dy}{dx} + 3y = e^{x}$ linear

$3 \to P(x)$, $e^{x} \to Q(x)$

② $y' - 2xy - \sin x = 0$ linear

$\dfrac{dy}{dx} - 2xy = \sin x$

$\dfrac{dy}{dx} + P(x)y = Q(x)$

$P(x) = -2x \qquad Q(x) = \sin x$

③ $\dfrac{dy}{dx} + xy = y^2$ ✗

Not linear

④ $\dfrac{dy}{dx} + e^{x}y = x^2$ linear

$e^{x} \to P(x)$, $x^2 \to Q(x)$

خطوات حل المعادلة

① نتأكد ان المعادلة خطية ونحدد قيمة P و Q

② نحسب معامل التكامل Integrating Factor

$$\rho(x) = e^{\int P(x)\,dx}$$

③ نضرب ρ بالمعادلة

$$\rho(x)\,y(x) = \int \rho(x)\,Q(x)\,dx + C$$

④ نكتب الحل على صورة

$$y(x) = \ldots\ldots$$

⑤ في حال توفر الشرط في السؤال نحسب قيمة الثابت C

مثال

$$y' + \frac{y}{x} = 3x$$

$$\frac{dy}{dx} + \frac{1}{x}\,y = 3x$$

$\frac{1}{x} \to P(x)$ $\qquad$ $3x \to Q(x)$

1

$P(x) = \frac{1}{x} \qquad Q(x) = 3x$

2

$$\rho(x) = e^{\int P(x)\,dx}$$

$$\rho(x) = e^{\int \frac{1}{x}\,dx} = \cancel{e}^{\cancel{\ln} x} = x$$

3

$$\rho(x)\, y(x) = \int Q(x)\,\rho(x)\,dx + C$$

$\downarrow$

$$x\,y = \int 3x \cdot x\,dx + C$$

$$x\,y = \int 3x^2\,dx + C$$

$$x\,y = x^3 + C$$

4

$$\boxed{y = x^2 + \frac{c}{x}}$$

Example 1 Solve the initial value problem

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \quad y(0) = -1.$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$ ①

$P(x) = -1 \qquad Q(x) = \frac{11}{8}e^{-x/3}$

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x}$$ ②

③

$$\rho y = \int \rho Q + C$$

$$e^{-x} y = \int e^{-x} \cdot \frac{11}{8} e^{-\frac{x}{3}} = \frac{11}{8}\int e^{-\frac{3}{3}x - \frac{x}{3}}$$

$$e^{-x} y = \frac{11}{8}\int e^{-\frac{4}{3}x}\,dx + C$$

$$e^{-x} y = \frac{11}{8}\,\frac{e^{-\frac{4}{3}x}}{-\frac{4}{3}} = -\frac{11 \cdot 3}{8 \cdot 4} e^{-\frac{4}{3}x} + C$$

$$e^{-x} y = \frac{-33}{32} e^{-\frac{4}{3}x} + C$$

نضرب كل المعادلة بـ e^{x} ④

$$y = -\frac{33}{32} e^{-\frac{4}{3}x} \cdot e^{\frac{3}{3}x} + Ce^{x}$$

$$\boxed{y = \frac{-33}{32} e^{-\frac{x}{3}} + Ce^{x}}$$

$y(0) = -1$ نطبق الشرط لحساب الثابت C (5)

$x = 0 \qquad y = -1$

$$-1 = \frac{-33}{32} + C$$

$$C = -1 + \frac{33}{32} = \frac{1}{32}$$

$$y = \frac{1}{32}e^{x} - \frac{33}{32}e^{-x/3}$$

$$y = \frac{1}{32}\left(e^{x} - 33e^{-x/3}\right)$$

Example 2 Find a general solution of

$$\frac{(x^2+1)}{x^2+1}\frac{dy}{dx} + \frac{3xy}{x^2+1} = \frac{6x}{x^2+1}. \qquad (9)$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

(1)

نقسم المعادلة على x^2+1

$$\frac{dy}{dx} + \underbrace{\frac{3x}{x^2+1}}_{P(x)}\,y = \underbrace{\frac{6x}{x^2+1}}_{Q(x)}$$

(2)

$$\rho(x) = e^{\int P(x)\,dx} = e^{\int \frac{3x}{x^2+1}\,dx}$$

$$\rho(x) = e^{\frac{3}{2}\ln|x^2+1|} = \cancel{e^{\ln}}(x^2+1)^{3/2}$$

$$\rho(x) = (x^2+1)^{3/2}$$

$$\rho y = \int Q\rho + c$$ ③

$$(x^2+1)^{3/2} y = \int \frac{6x}{(x^2+1)^1}(x^2+1)^{3/2}\,dx$$

$$(x^2+1)^{3/2} y = \int 6x\,(x^2+1)^{\frac{1}{2}}\,dx$$

$u = x^2+1 \qquad \frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}$

$$(x^2+1)^{3/2} y = \int 6\cancel{x}\,u^{1/2}\,\frac{du}{2\cancel{x}}$$

$$(x^2+1)^{3/2} y = 3\int u^{1/2}\,du = 2u^{3/2} + c$$

$$(x^2+1)^{3/2} y = 2(x^2+1)^{3/2} + c$$

نقسم على $(x^2+1)^{3/2}$ ④

$$\boxed{y = 2 + c(x^2+1)^{-3/2}}$$

Example 3 Solve the initial value problem

$x_0 = 1$

$$x^2\frac{dy}{dx} + xy = \sin x, \quad y(1) = y_0. \tag{13}$$

نقسم على x^2

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$\frac{dy}{dx} + \frac{1}{x}\,y = \frac{\sin x}{x^2} \qquad (1)$$

$P(x)$ $\qquad$ $Q(x)$

$$\rho = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}dx} = \cancel{e}^{\cancel{\ln} x} = x \qquad (2)$$

$$(3)$$

$$y(x) = \frac{1}{\rho(x)}\left[y_0 + \int_{x_0}^{x} \rho(x)\,Q(x)\,dx\right]$$

$$y(x) = \frac{1}{x}\left[y_0 + \int_{1}^{x} \frac{\sin x}{x}\,dx\right]$$

$$y(x) = \frac{1}{x}\left[y_0 + Si(x) - Si(1)\right]$$

1.5 Problems

Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.

1. $y' + y = 2,\ y(0) = 0$

(1) $$\frac{dy}{dx} + \underset{P(x)}{1}\, y = \underset{Q(x)}{2}$$

$P(x) = 1 \qquad Q(x) = 2$

(2) $$\rho = e^{\int P(x)\,dx} = e^{\int dx} = e^{x}$$

(3) $$\rho(x)\, y(x) = \int \rho(x)\, Q(x)\, dx$$

$$e^{x}\, y(x) = \int 2e^{x}\, dx$$

$$e^{x}\, y(x) = 2e^{x} + C$$

(4) $$y(x) = 2 + Ce^{-x}$$

(5) $y(0) = 0 \qquad 0 = 2 + C\cancel{e^{0}} \qquad \boxed{C = -2}$

$$y(x) = 2 - 2e^{-x}$$

2. $y' - 2y = 3e^{2x}$, $y(0) = 0$

$$\frac{dy}{dx} - 2y = 3e^{2x}$$

① $P(x) = -2$ $\quad Q = 3e^{3x}$

② $\rho(x) = e^{\int p(x)} = e^{\int -2dx} = e^{-2x}$

③ $\rho y = \int \rho Q$

$$e^{-2x} y = \int e^{-2x} \cdot 3e^{2x} \, dx$$

$$e^{-2x} y = 3\int dx$$

④ $\left(e^{-2x} y = 3x + c \right) \cdot e^{2x}$

$$\boxed{y = 3xe^{2x} + ce^{2x}}$$

⑤ $0 = \cancel{3(0)e^{0}} + c\cancel{e^{0}}$

$$c = 0$$

$$\boxed{y = 3xe^{2x}}$$

3. $y' + 3y = 2xe^{-3x}$

$$\frac{dy}{dx} + 3y = 2xe^{-3x}$$

① $P(x) = 3$ $\quad Q(x) = 2xe^{-3x}$

② $\rho = e^{\int 3\,dx} = e^{3x}$

③ $\rho y = \int \rho Q \, dx$

$$e^{3x}y = \int \cancel{e^{3x}} \cdot 2x\, \cancel{e^{-3x}} \, dx$$

$$e^{3x}y = x^2 + C$$

$$y = \frac{x^2}{e^{3x}} + \frac{C}{e^{3x}}$$

$$\boxed{y = e^{-3x}(x^2 + C)}$$

4. $y' - 2xy = e^{x^2}$

$$\frac{dy}{dx} - 2xy = e^{x^2}$$

① $P(x) = -2x$ $\qquad Q(x) = e^{x^2}$

② $$\rho = e^{\int P(x)\,dx} = e^{\int -2x} = e^{-x^2}$$

③ $$\rho y = \int Q\rho \, dx$$

$$e^{-x^2} y = \int \cancel{e^{x^2} \cdot e^{-x^2}} \, dx$$

$$e^{-x^2} y = \int dx$$

$$e^{-x^2} y = x + C$$

$$\boxed{y = e^{x^2}(x + C)}$$

9. $xy' - y = x,\ y(1) = 7$

$$xy' - y = x \implies y' - \frac{1}{x}y = 1$$

① $$\frac{dy}{dx} - \frac{1}{x}y = 1 \quad \begin{cases} P(x) = -\frac{1}{x} \\ Q(x) = 1 \end{cases}$$

② $$\rho = e^{\int -\frac{1}{x}} = \cancel{e}^{\cancel{\ln} x^{-1}} = x^{-1} = \frac{1}{x}$$

③ $$\rho y = \int \rho Q\, dx$$

$$\frac{1}{x}y = \int \frac{1}{x}\, dx$$

$$\frac{1}{x}y = \ln x + C$$

④ $$y = x\ln x + xC$$

⑤ $$7 = 1\cancel{\ln 1} + 1C$$

$$c = 7$$

$$\boxed{y = x\ln x + 7x}$$

10. $\dfrac{2xy'}{2x} - \dfrac{3y}{2x} = \dfrac{9x^3}{2x}$

① $\dfrac{dy}{dx} - \dfrac{3}{2x} y = \dfrac{9}{2} x^2$

$P(x) = -\dfrac{3}{2x}$

$Q(x) = \dfrac{9}{2} x^2$

② $\rho = e^{\int P(x)} = e^{-\frac{3}{2}\int \frac{1}{x}\, dx} = e^{-\frac{3}{2}\ln x} = \cancel{e}^{\cancel{\ln} x^{-3/2}} = x^{-3/2}$

③ $\rho y = \int \rho Q \, dx$

$-\frac{3}{2} + \frac{4}{2}$

$x^{-3/2} y = \int x^{-3/2} \cdot \dfrac{9}{2} x^2 \, dx$

$x^{-3/2} y = \dfrac{9}{2} \int x^{1/2} dx = \dfrac{\cancel{9}^{3}}{\cancel{2}} \dfrac{x^{3/2}}{\frac{\cancel{3}}{\cancel{2}}} = 3x^{3/2} + C$

$x^{-3/2} y = 3x^{3/2} + C$

$y = 3x^{3/2} x^{3/2} + C x^{3/2}$

$$\boxed{y = 3x^3 + C x^{3/2}}$$

11. $xy' + y = 3xy,\ y(1) = 0$

$\frac{xy'}{x} + \frac{y}{x} = \frac{3xy}{x}$

(1) $\frac{dy}{dx} + \frac{1}{x} y = 3y$

$\frac{dy}{dx} + P(x) y = Q(x)$

$\frac{dy}{dx} + \frac{1}{x} y - 3y = 0$

$\frac{dy}{dx} + \underbrace{\left(\frac{1}{x} - 3\right)}_{P(x)} y = \underbrace{0}_{Q(x)}$

(2) $\rho = e^{\int P(x)\,dx}$

$= e^{\int \left(\frac{1}{x} - 3\right) dx} = e^{\ln x - 3x}$

(3) $\rho y = \int \rho Q \, dx$

$y e^{\ln x - 3x} = \int (e^{\ln x - 3x}) 0 \, dx$

$y e^{\ln x - 3x} = C$

$y = C \cancel{e}^{\ln x^{-1}} e^{3x}$

$y = \frac{C e^{3x}}{x}$

$0 = \frac{C e^{3}}{1}$

$C = 0$

$$\boxed{y = 0}$$

12. $xy' + 3y = 2x^5$, $y(2) = 1$

$\overline{x} \qquad \overline{x} \qquad \overline{x}$

① $y' + \frac{3}{x} y = 2x^4$

$P(x) = \frac{3}{x}$

$Q(x) = 2x^4$

② $\rho = e^{\int P(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{\ln x^3} = x^3$

③ $\rho y = \int \rho Q \, dx$

$$x^3 y = 2\int x^7 \, dx$$

$$x^3 y = \frac{1}{4} x^8 + C$$

$$\boxed{y = \frac{1}{4} x^5 + C x^{-3}}$$

$y(2) = 1 \qquad x = 2 \qquad y = 1$

$$1 = \frac{1}{4}(2)^5 + \frac{C}{2^3}$$

$$1 = 8 + \frac{C}{8} \Rightarrow \frac{C}{8} = -7$$

$$\boxed{C = -56}$$

$$y = \frac{1}{4} x^5 - 56 x^{-3}$$

19. $y' + y\cot x = \cos x$

① $\frac{dy}{dx} + \cot x \, y = \cos x$ $\rightarrow P(x) = \cot x$, $\rightarrow Q(x) = \cos x$

② $\rho = e^{\int \cot x} = e^{\cancel{\ln} \sin x}$

$\int \frac{\cos x}{\sin x} = \int \frac{f'}{f}$

$\ln f = \ln(\sin x)$

$\rho = \sin x$

③ $y\rho = \int \rho \, Q(x)$

$y \sin x = \int \underline{\sin x} \cos x \, dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$y \sin x = \int u \cancel{\cos x} \frac{du}{\cancel{\cos x}}$

$y \sin x = \frac{u^2}{2} + C$

$y \sin x = \frac{\sin^2 x}{2} + C$

④ $\boxed{y = \frac{\sin x}{2} + \frac{C}{\sin x}}$

$y = \frac{\sin x}{2} + C \csc x$

Exact DE

$$\underbrace{M(x,y)}\,dx + \underbrace{N(x,y)}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{Exact}$$

لتأكد ان المعادله هي Exact ولا

① نكتب المعادله على صوره $M(x,y)dx + N(x,y)dy = 0$

② نجب $\frac{\partial M}{\partial y}$ و $\frac{\partial N}{\partial x}$ اذا كانا متساوين

تكون المعادله Exact

مثال ٤ اثبت ان المعادله Exact ثم حلها

$$(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

$$\underbrace{(2xy - 9x^2)}_{M}\,dx + \underbrace{(2y + x^2 + 1)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{set exact}$$

طريقة حل المعادلة التفاضلية من نوع Exact

① نكتب المعادلة على صيغة $M\,dx + N\,dy = 0$

② نتأكد ان المعادلة Exact $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

③ نكامل كل من الحدين

$$\int M\,dx = \text{.........} + \cancel{g(y)} \rightarrow c$$

$$\int N\,dy = \text{.........}$$

④ نجمع النتيجتين بدون تكرار

$$\underline{\qquad\qquad\qquad\qquad} = C$$

$$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$$

$$\int M\,dx = \int (2xy - 9x^2)\,dx = \cancel{\tfrac{2}{2}}x^2y - 3x^3 + g(y)$$

$$\int N\,dy = \int (2y + x^2 + 1)\,dy = y^2 + x^2y + y + c$$

$$\boxed{x^2y - 3x^2 + y^2 + y = C}$$

Example 8 The differential equation is exact or not

$$y^3\,dx + 3xy^2\,dy = 0$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M(x,y) = y^3$ $\qquad$ $N(x,y) = 3xy^2$

$\dfrac{\partial M}{\partial y} = 3y^2$ $\qquad$ $\dfrac{\partial N}{\partial x} = 3y^2$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Exact

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$$y^3\,dx + 3xy^2\,dy = 0$$

القسمة على y وإيجاد عامل مشترك غير صحيحة

$$y\,dx + 3x\,dy = 0$$

$\dfrac{\partial M}{\partial y} = 1$ $\qquad$ $\dfrac{\partial N}{\partial x} = 3$

Not exact

بعد القسمة تتغير المعادلة و شرط Exact لا يكون صحيح

Example 9 Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0.$$

1

$$(\underbrace{6xy - y^3}_{M})\,dx + (\underbrace{4y + 3x^2 - 3xy^2}_{N})\,dy = 0$$

2

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 6x - 3y^2 \\ \frac{\partial N}{\partial x} &= 6x - 3y^2 \end{aligned}\right\} = \text{Exact}$$

3

$$\int (6xy - y^3)\,dx = 3x^2y - y^3x + g(y)$$

$$\int (4y + 3x^2 - 3xy^2)\,dy = 2y^2 + 3x^2y - xy^3 + c$$

$$F(x,y) = 3x^2y - y^3x + 2y^2 + C_1$$

$$\boxed{3x^2y - y^3x + 2y^2 = c}$$

1.6 Problems

In Problems 31 through 42, verify that the given differential equation is exact; then solve it.

31. $\underbrace{(2x + 3y)}_{M}\,dx + \underbrace{(3x + 2y)}_{N}\,dy = 0$

$M = 2x + 3y \qquad N = 3x + 2y$

$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact equation

$\int (2x + 3y)\,dx = x^2 + 3yx + g(y)$

$\int (3x + 2y)\,dy = 3xy + y^2 + C_1$

$$\boxed{x^2 + y^2 + 3xy = C}$$

32. $(4x - y)\,dx + (6y - x)\,dy = 0$

$M = 4x - y$ $\qquad$ $N = 6y - x$

$\dfrac{\partial M}{\partial y} = -1 \quad = \quad \dfrac{\partial N}{\partial x} = -1$ $\qquad$ Exact

$$\int (4x - y)\,dx = 2x^2 - yx + g(y)$$

$$\int (6y - x)\,dy = 3y^2 - xy + c_1$$

$$\boxed{2x^2 + 3y^2 - xy = c}$$

33. $(3x^2 + 2y^2)\,dx + (4xy + 6y^2)\,dy = 0$

M $\quad$ N

$$\frac{\partial M}{\partial y} = 4y \quad = \quad \frac{\partial N}{\partial x} = 4y$$

exact

$$\int (3x^2 + 2y^2)\,dx = x^3 + 2y^2x + g(y)$$

$$\int (4xy + 6y^2)\,dy = 2xy^2 + 2y^3 + C_1$$

$$\boxed{x^3 + 2y^2x + 2y^3 = C}$$

34. $(2xy^2 + 3x^2)\,dx + (2x^2y + 4y^3)\,dy = 0$

$M = 2xy^2 + 3x^2 \qquad\qquad N = 2x^2y + 4y^3$

$$\frac{\partial M}{\partial y} = 4xy \quad = \quad \frac{\partial N}{\partial x} = 4xy$$

exact

$$\int (2xy^2 + 3x^2)\,dx = x^2y^2 + x^3 + g(y)$$

$$\int (2x^2y + 4y^3)\,dy = x^2y^2 + y^4 + C_1$$

$$\boxed{x^3 + y^4 + x^2y^2 = C}$$

35. $\left(x^3 + \frac{y}{x}\right) dx + (y^2 + \ln x)\, dy = 0$

$M = x^3 + \frac{y}{x}$ $\qquad$ $N = y^2 + \ln x$

$\frac{\partial M}{\partial y} = \frac{1}{x}$ $\qquad$ $\frac{\partial N}{\partial x} = \frac{1}{x}$

Exact

$$\int \left(x^3 + \frac{y}{x}\right) dx = \frac{x^4}{4} + y \ln x + g(y)$$

$$\int (y^2 + \ln x)\, dy = \frac{y^3}{3} + y \ln x + C_1$$

$$\frac{x^4}{4} + \frac{y^3}{3} + y \ln x = C$$

36. $(1 + ye^{xy})\,dx + (2y + xe^{xy})\,dy = 0$

$$M = 1 + ye^{xy} \qquad N = 2y + xe^{xy}$$

$$\frac{\partial M}{\partial y} = yxe^{xy} + e^{xy} \qquad \frac{\partial N}{\partial x} = xye^{xy} + e^{xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies \text{exact}$$

$$\int e^{3x} = \frac{e^{x}}{3}$$

$$\int (1 + ye^{xy})\,dx = x + e^{xy} + g(y)$$

$$\int (2y + xe^{xy})\,dy = y^2 + \frac{\cancel{x}\,e^{xy}}{\cancel{x}} + C_1$$

$$\boxed{x + y^2 + e^{xy} = c}$$

39. $(3x^2y^3 + y^4)\,dx + (3x^3y^2 + y^4 + 4xy^3)\,dy = 0$

$M = 3x^2y^3 + y^4$ $\qquad$ $N = 3x^3y^2 + y^4 + 4xy^3$

$\dfrac{\partial M}{\partial y} = 9x^2y^2 + 4y^3$ $\qquad$ $\dfrac{\partial N}{\partial x} = 9x^2y^2 + 4y^3$

exact

$$\int (3x^2y^3 + y^4)\,dx = x^3y^3 + xy^4 + G(y)$$

$$\int (3x^3y^2 + y^4 + 4xy^3)\,dy = x^3y^3 + \frac{y^5}{5} + xy^4$$

$$\boxed{x^3y^3 + xy^4 + \frac{y^5}{5} = C}$$

40. $\underbrace{(e^x \sin y + \tan y)}_{M}\, dx + \underbrace{(e^x \cos y + x \sec^2 y)}_{N}\, dy = 0$

$M = e^x \sin y + \tan y$ $\qquad$ $N = e^x \cos y + x \sec^2 y$

$\dfrac{\partial M}{\partial y} = e^x \cos y + \sec^2 y$ $\qquad$ $\dfrac{\partial N}{\partial x} = e^x \cos y + \sec^2 y$

$\Rightarrow$ exact

$$\int (e^x \sin y + \tan y)\, dx = e^x \sin y + x \tan y + G(y)$$

$$\int (e^x \cos y + x \sec^2 y)\, dy = e^x \sin y + x \tan y$$

$$e^x \sin y + x \tan y = C$$