CHAPTER 11 Geometry in Space and Vectors

11.1 Cartesian Coordinates in Three-Space

(x, y, z)

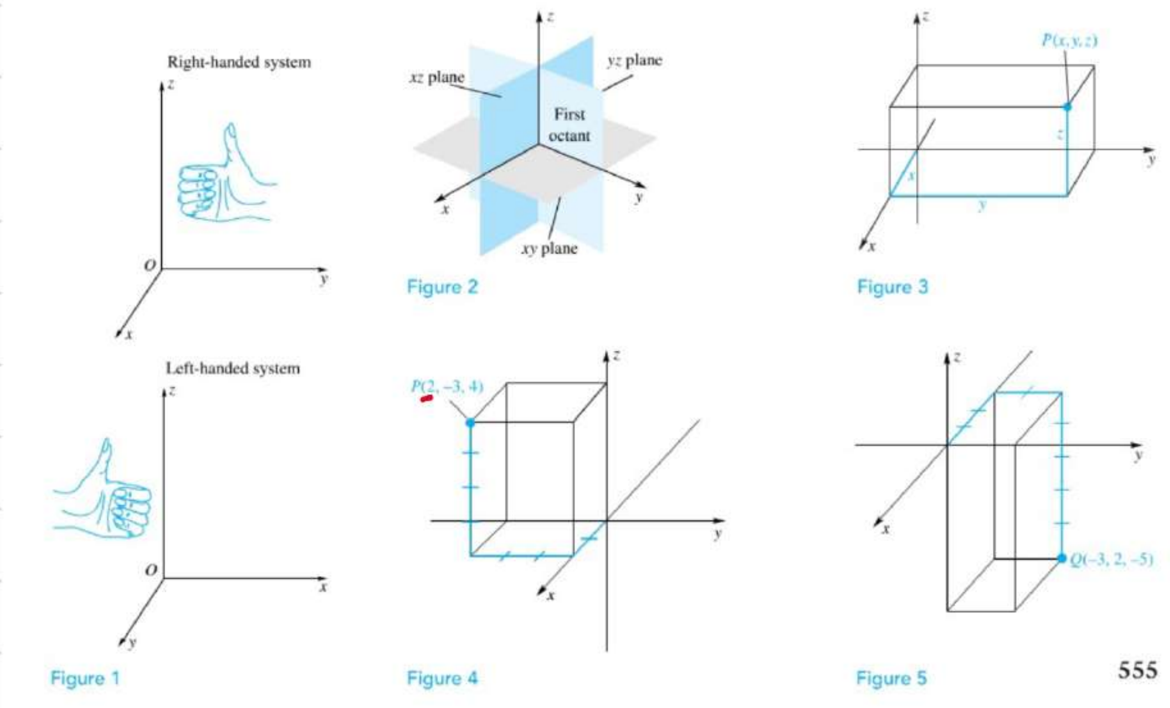
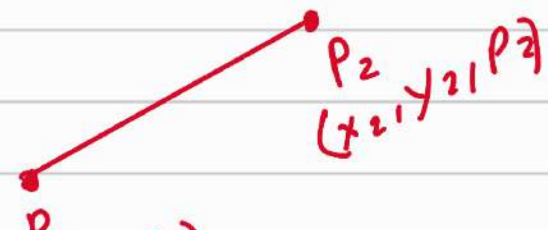


Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

555

z

المسافة بين نقطتين

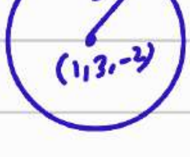


$$d = |P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

معادلة الدائره الكرة

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

(h, k, l) مركز الكرة $r =$ نصف قطرها



$$(x-1)^2 + (y-3)^2 + (z+2)^2 = 9$$

EXAMPLE 1 Find the distance (مسافة) between the points $P(2, -3, 4)$ and $Q(-3, 2, -5)$, which were plotted in Figures 4 and 5.

$x_1 \quad y_1 \quad z_1$

$x_2 \quad y_2 \quad z_2$

$$|PQ| = \sqrt{(-3-2)^2 + (2--3)^2 + (-5-4)^2}$$

$$\sqrt{25 + 25 + 81} = \sqrt{131}$$

EXAMPLE 2 Find the center and radius of the sphere with equation

$$x^2 + y^2 + z^2 - 10x - 8y - 12z + 68 = 0$$

$+\left(\frac{b}{2}\right)^2$

تحويل المعادلة الى الشكل العام باكمال المربع

$$(x^2 - 10x + 25) + (y^2 - 8y + 16) + (z^2 - 12z + 36) = -68 + 25 + 16 + 36$$

تحويل كل المعادلات الى مربع كامل

$$(x-5)^2 + (y-4)^2 + (z-6)^2 = 9$$

centre $= (5, 4, 6)$ $\qquad r = 3$

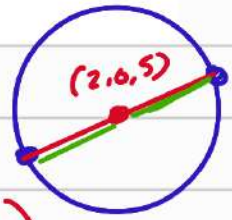
Midpoint

احداثيات نقطة المنتصف

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

EXAMPLE 3 Find the equation of the sphere that has the line segment joining $(-1, 2, 3)$ and $(5, -2, 7)$ as a diameter (Figure 9).

مركز الكرة يساوي نقطة المنتصف للنقطتين



$$\text{Centre} = \left(\frac{-1+5}{2}, \frac{2+-2}{2}, \frac{3+7}{2} \right)$$

$$(2, 0, 5)$$

حساب نصف القطر نحسب المسافة من المركز الى احد النقطتين

$$(2,0,5) \quad (5,-2,7) \qquad r^2 = 17$$

$$r = \sqrt{3^2 + (-2)^2 + (2)^2} = \sqrt{17}$$

نكتب معادلة الدائرة

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$$(x-2)^2 + (y)^2 + (z-5)^2 = 17$$

EXAMPLE 4 Sketch the graph of $3x + 4y + 2z = 12$.

معادلة خطية

للرسم البياني نحسب التقاطع

X-intercept $\Rightarrow$ نعوض كل الباقي صفر

$$3x + 0 + 0 = 12 \Rightarrow 3x = 12$$

$\boxed{x = 4}$ $\boxed{y = 0}$ $\boxed{z = 0}$

y-intercept نعوض كل الباقي صفر

$$0 + 4y + 0 = 12 \qquad 4y = 12$$

$\boxed{y=3}$ $\boxed{x=0}$ $\boxed{z=0}$

z-intercept

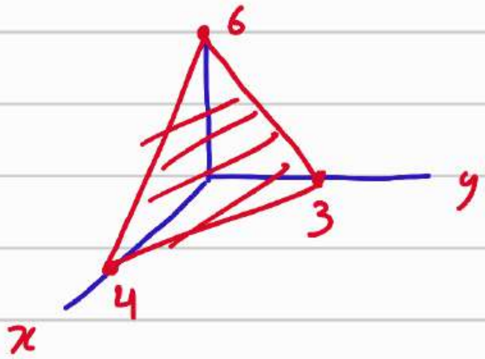
$2z = 12$ $z = 6$

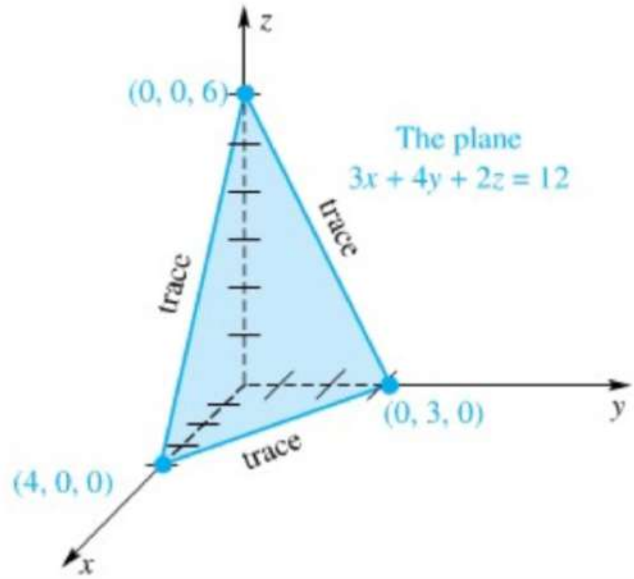
$\boxed{z=6}$ $\boxed{x=0}$ $\boxed{y=0}$

$(4,0,0)$

$(0,3,0)$

$(0,0,6)$





EXAMPLE 6 An object's position at time t is given by the parametrically defined curve $x = \cos t$, $y = \sin t$, $z = t/\pi$ for $0 \le t \le 2\pi$. Sketch this curve and find its arc length.

arc length

$$\boxed{\ell = \int_a^b \sqrt{f'^2 + g'^2 + h'^2}\, dt}$$ طول القوس

$$\ell = \int_0^{2\pi} \sqrt{(-\sin t)^2 + \cos^2 t + \frac{1}{\pi^2}}\, dt$$

$$\mathcal{L} = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + \frac{1}{\pi^2}}\, dt$$

$$\mathcal{L} = \int_0^{2\pi} \sqrt{1 + \frac{1}{\pi^2}}\, dt = \sqrt{1 + \frac{1}{\pi^2}}\; t \Big]_0^{2\pi}$$

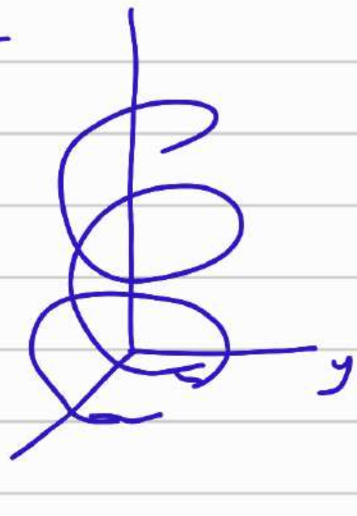
$$\mathcal{L} = 2\pi \sqrt{1 + \frac{1}{\pi^2}}$$

$x = \cos t$ $\qquad$ $y = \sin t$

معادله دایره

$y = \frac{1}{\pi} t$

خط مستقیم



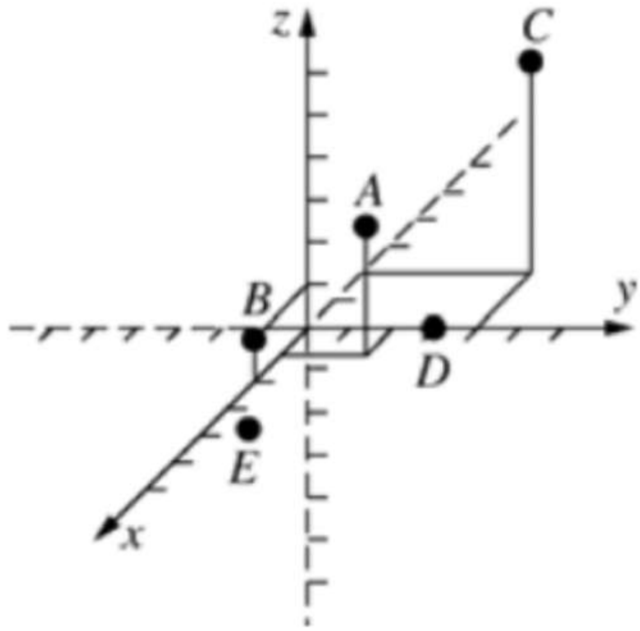
Helix

Problem Set 11.1

1. Plot the points whose coordinates are $(1, 2, 3)$, $(2, 0, 1)$, $(-2, 4, 5)$, $(0, 3, 0)$, and $(-1, -2, -3)$. If appropriate, show the "box" as in Figures 4 and 5.

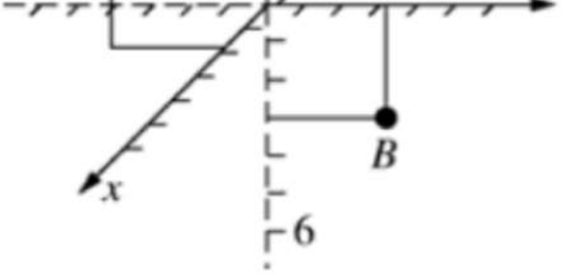
2. Follow the directions of Problem 1 for $\left(\sqrt{3}, -3, 3\right)$, $(0, \pi, -3)$, $\left(-2, \frac{1}{3}, 2\right)$, and $(0, 0, e)$.

1. $A(1, 2, 3)$, $B(2, 0, 1)$, $C(-2, 4, 5)$, $D(0, 3, 0)$, $E(-1, -2, -3)$

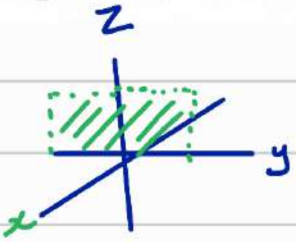


y

2. $A\left(\sqrt{3}, -3, 3\right)$, $B(0, \pi, -3)$, $C\left(-2, \dfrac{1}{3}, 2\right)$, $D(0, 0, e)$



3. What is peculiar to the coordinates of all points in the yz-plane? On the z-axis?



yz-plane

$x = 0$

$x = 0$

z-axis $\Rightarrow$ $y = 0$

5. Find the distance between the following pairs of points.

(a) $(6, -1, 0)$ and $(1, 2, 3)$

$x_1 \; y_1 \; z_1 \qquad x_2 \; y_2 \; z_2$

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

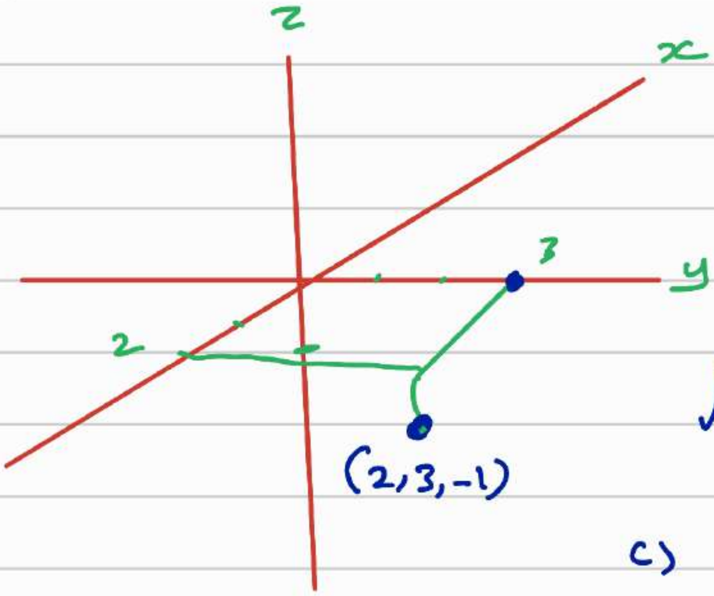
$$= \sqrt{(6-1)^2 + (-1-2)^2 + (0-3)^2}$$

$$= \sqrt{25 + 9 + 9} = \sqrt{43}$$

8. Find the distance from $(2, 3, -1)$ to

(a) the xy-plane, (b) the y-axis, and

(c) the origin.



a) 1

b) $(2,3,-1)$ $(0,3,0)$

$$\sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$$

c) $(2, 3, -1)$ $(0,0,0)$

$$\sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

11. Write the equation of the sphere with the given center and radius.

(a) $(1,2,3); 5$ (b) $(-2,-3,-6); \sqrt{5}$

r

(c) $(\pi, e, \sqrt{2}); \sqrt{\pi}$

$x_1 \quad y_1 \quad z_1$

معادلة الدائرة

$$(x-x_1)^2+(y-y_1)^2=r^2$$

معادلة الكرة

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2$$

a) $(x-1)^2+(y-2)^2+(z-3)^2=25$

b) $(x+2)^2+(x+3)^2+(z+6)^2=5$

c) $(x-\pi)^2+(y-e)^2+(z-\sqrt{2})^2=\pi$

In Problems 13–16, complete the squares to find the center and radius of the sphere whose equation is given (see Example 2).

13. $x^2+y^2+z^2-12x+14y-8z+1=0$

$$\underbrace{(x^2-12x+6^2)}+\underbrace{(y^2+14y+7^2)}+\underbrace{(z^2-8z+4^2)}=-1+6^2+7^2+4^2$$

$$(x-6)^2+(y+7)^2+(z-4)^2=10^2$$

$r=10$ $(6, -7, 4)$

15. $4x^2 + 4y^2 + 4z^2 - 4x + 8y + 16z - 13 = 0$

$$x^2 + y^2 + z^2 - x + 2y + 4z = \frac{13}{4}$$

$$\left(x^2 - x + \left(\frac{1}{2}\right)^2\right) + \left(y^2 + 2y + 1^2\right) + \left(z^2 + 4z + 2^2\right) = \frac{13}{4} + \frac{1}{4} + 1 + 4$$

$$\left(x - \frac{1}{2}\right)^2 + (y+1)^2 + (z+2)^2 = \frac{13}{4} + \frac{1}{4} + \frac{4}{4} + \frac{16}{4}$$

$$\left(x - \frac{1}{2}\right)^2 + (y+1)^2 + (z+2)^2) = \frac{17}{2}$$

$r = \sqrt{\frac{17}{2}}$ center $= \left(\frac{1}{2}, -1, -2\right)$

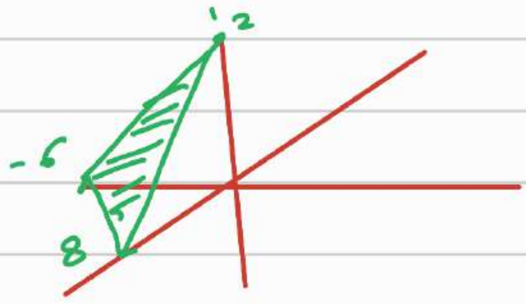
In Problems 17–24, sketch the graphs of the given equations. Begin by sketching the traces in the coordinate planes (see Examples 4 and 5).

18. $3x - 4y + 2z = 24$

x- intercept $y=0$ $z=0$ $3x = 24$ $\boxed{x = 8}$

y- intercept $x=0$ $z=0$ $-4y = 24$ $\boxed{y = -6}$

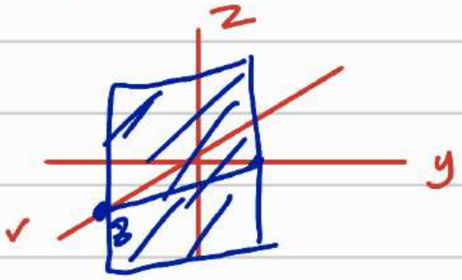
z- intercept $y=0$ $x=0$ $2z = 24$ $\boxed{z = 12}$

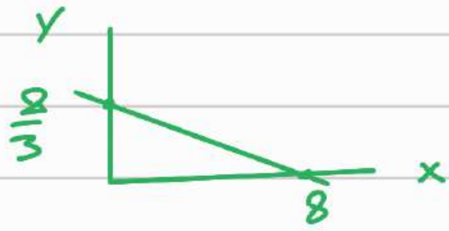


21. $x + 3y = 8$

x-intercept $\quad y=0 \quad x=8$

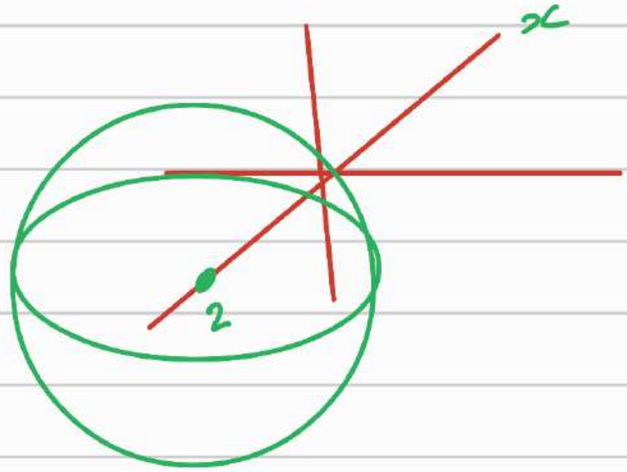
y-interept $\quad x=0 \quad 3y=8 \quad y=\frac{8}{3}$





24. $(x-2)^2 + y^2 + z^2 = 4$ $\quad$ معادلة كرة

$r=2$ $\quad$ centre $:(2,0,0)$



In Problems 25–32, find the arc length of the given curve.

27. $\overset{f}{x} = t^{3/2}, \overset{g}{y} = 3t, \overset{h}{z} = 4t; 1 \leq t \leq 4$

$$L = \int \sqrt{f'^2 + g'^2 + h'^2}\, dt$$

$$f' = \frac{3}{2}\sqrt{t} \qquad g' = 3 \qquad h' = 4$$

$$L = \int \sqrt{\frac{9}{4}t + 9 + 16}\, dt = \int_1^4 \sqrt{\frac{9}{4}t + 25}\, dt$$

$$\int_1^4 \left(\frac{9}{4}t + 25\right)^{\frac{1}{2}} dt$$

$$\frac{8}{27}\left(\frac{9}{4}t + 25\right)^{3/2}\Big]_1^4$$

$$\frac{8}{27}\left[(9+25)^{3/2} - \left(\frac{9}{4}+25\right)^{3/2}\right]$$

$$\frac{8}{27}\left[(34)^{3/2} - \left(\frac{109}{4}\right)^{3/2}\right] = 16.59$$

$\int x^2\, dx$	$\frac{x^3}{3}$
$\int x^{\frac{1}{2}}$	$\frac{2}{3}x^{3/2}$
$\int (3x+7)^4$	$\frac{(3x+7)^5}{15}$
$\int (9x+2)^{\frac{1}{2}}$	$2\frac{(9x+2)^{3/2}}{27}$

29. $x = t^2, y = (4/3)t^{3/2}, z = t; 0 \leq t \leq 8$

$f' = 2t \qquad g' = \frac{4}{\cancel{3}}\frac{\cancel{3}}{2} t^{\frac{1}{2}} = 2\sqrt{t} \qquad h' = 1$

$$d = \int_0^8 \sqrt{4t^2 + 4t + 1}\, dt$$

$$d = \int_0^8 \sqrt{(2t+1)^{\cancel{2}}}\, dt$$

$$d = \int_0^8 2t + 1 \quad dt$$

$$\cancel{\frac{2}{2}} t^2 + t \Big]_0^8$$

$$(64 + 8) - 0 = 72$$

11.2
Vectors

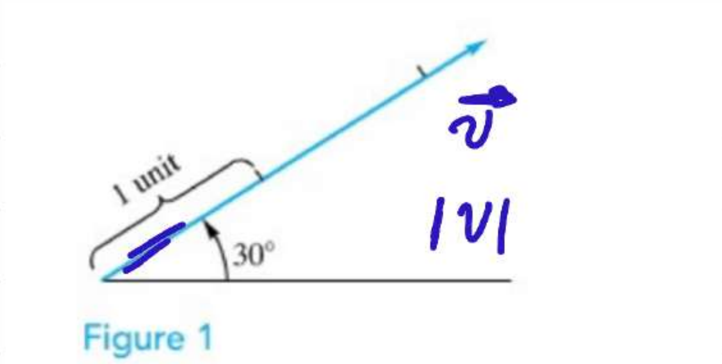


Figure 1

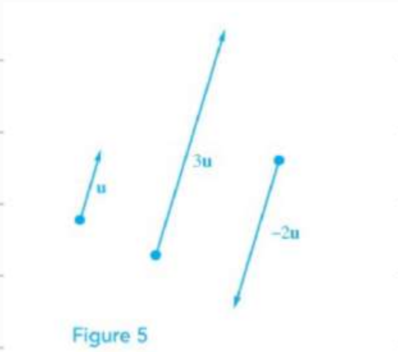
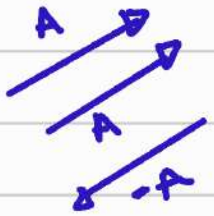


Figure 5

Scalar ⇒ قياسية   magnitude

Vector ⇒ متجه   magnitude   direction

الطول يدل على المقدار
الميله محور × يدل على الاتجاه



تتساوي المتجهات اذا كان المقدار والاتجاه نفسه
المتجهات متعاكسة عندما يكون نفس المقدار لكن
عكس الاتجاه

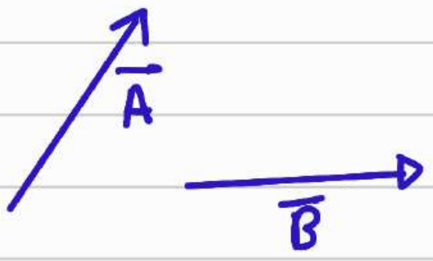
المتجه الوحدة (u) :- طوله (1) وبنفس اتجاه المتجه الاصلي

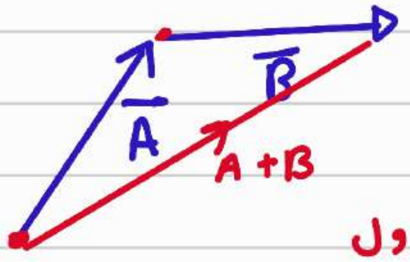
متجه الوحدة يخص
ب B

Operations on vectors

Sum ⇒ resultant   مجموع المحصلة

طريقة المثلث

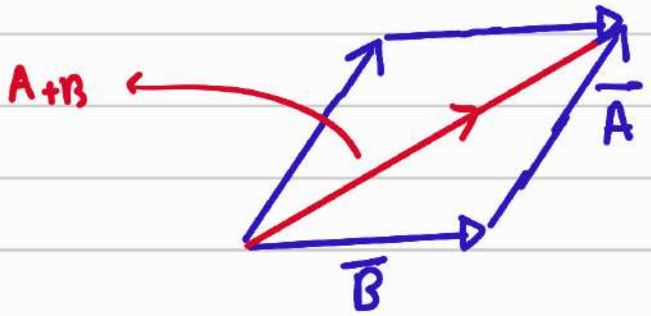




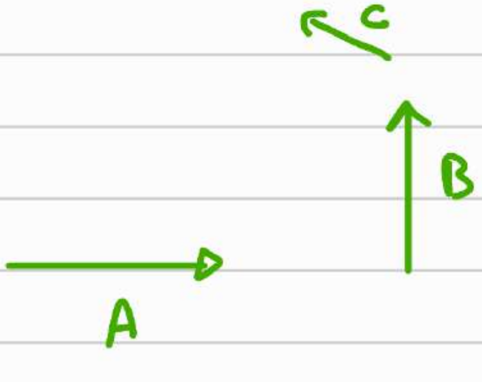
$\vec{A} + \vec{B}$

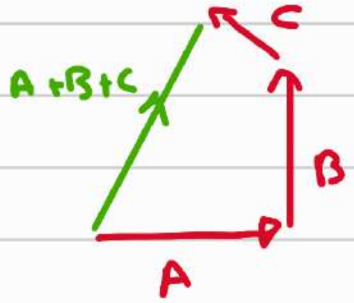
الجمع من ذيل الاول
راس الثاني

طريقة متوازي الاضلاع



جمع 3 متجهات





EXAMPLE 1 In Figure 6, express **w** in terms of **u** and **v**.

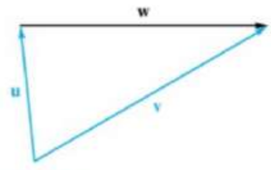


Figure 6

$$u + w = v$$

$$\boxed{w = v - u}$$

EXAMPLE 2 In Figure 7, $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{AC}$. Express **m** in terms of **u** and **v**.

$$m = u + AB \Rightarrow \boxed{m = u + \frac{2}{3}AC}$$

$$\boxed{v = u + AC}$$

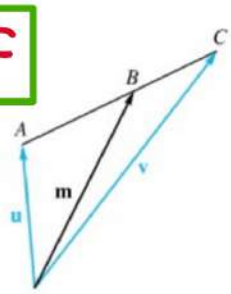


Figure 7

نعوض AC في المعادله الاولى

$$AC = v - u$$

تصبح المعادله الاولى

$$m = u + \frac{2}{3}(v - u) = u + \frac{2}{3}v - \frac{2}{3}u$$

$$\boxed{m = \frac{1}{3}u + \frac{2}{3}v}$$

حاله عامة $AB = t\,AC$

$$m = u + t(v - u)$$

$$m = u + tv - tu.$$

$$\boxed{m = u(1 - t) + tv}$$

العمليات على المتجهات

Theorem A

For any vectors **u**, **v**, and **w**, and any scalars a and b, the following relationships hold.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ تبديل
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ تجميعية
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ محايد جمعي
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ نظير جمعي
5. $a(b\mathbf{u}) = (ab)\mathbf{u}$ الضرب بعدد
6. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ توزيع
7. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ توزيع
8. $1\mathbf{u} = \mathbf{u}$ نظير الضرب
9. $\|a\mathbf{u}\| = |a|\|\mathbf{u}\|$

$A = \langle 3,4 \rangle \qquad B = \langle 6,8,1 \rangle \qquad C = \langle 1,1 \rangle$

* $\|A\| = \sqrt{3^2+4^2} = 5$

طول المتجه مقدار

* $\|B\| = \sqrt{6^2+8^2+1^2} = \sqrt{101}$

* $2A = 2\langle 3,4 \rangle = \langle 6,8 \rangle$

* $A+C = \langle 3,4 \rangle + \langle 1,1 \rangle = \langle 4,5 \rangle$

* $A - 2C = \langle 3,4 \rangle - 2\langle 1,1 \rangle$

$= \langle 3,4 \rangle - \langle 2,2 \rangle$

$A-2C = \langle 1,2 \rangle$

EXAMPLE 4 Let $\mathbf{u} = \langle 1, 1, 2 \rangle$ and $\mathbf{v} = \langle 0, -1, 2 \rangle$. Find (a) $\mathbf{u} + \mathbf{v}$, and (b) $\mathbf{u} - 2\mathbf{v}$, and express them in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Find (c) $\|\mathbf{u}\|$, and (d) $\|-3\mathbf{u}\|$.

a) $u + v = \langle 1, 1, 2 \rangle + \langle 0, -1, 2 \rangle$

$\langle 1, 0, 4 \rangle = i + 4k$

b) $u - 2v = \langle 1, 1, 2 \rangle - 2\langle 0, -1, 2 \rangle$

$= \langle 1, 1, 2 \rangle - \langle 0, -2, 4 \rangle$

$\langle 1, 3, -2 \rangle$

$= i + 3j - 2k$

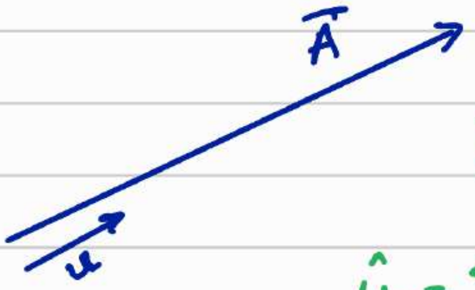
c) $|u| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

d) $\|-3u\| = |-3| \, \|u\| = 3\sqrt{6}$

(unit vector) $\hat{u}$ متجه الوحدة

متجه طوله 1 وله نفس اتجاه المتجه الاصلي

$\hat{u} = \dfrac{\text{المتجه}}{\text{طوله}} = \dfrac{\vec{A}}{|A|}$



$\vec{A} = \langle 1, 2 \rangle$ اوجد متجه الوحده للمتجه

$\hat{u} = \dfrac{\langle 1, 2 \rangle}{\sqrt{1^2 + 2^2}} = \dfrac{\langle 1, 2 \rangle}{\sqrt{5}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

EXAMPLE 5 Let $\mathbf{v} = \langle 4, -3 \rangle$. Find $\|\mathbf{v}\|$, and find a unit vector $\mathbf{u}$ with the same direction as $\mathbf{v}$.

$$\|V\| = \sqrt{4^2 + (-3)^2} = \sqrt{16+3} =$$

$$\sqrt{25} = 5$$

$$\hat{u} = \frac{V}{\|V\|} = \frac{\langle 4, -3 \rangle}{5}$$

$$\hat{u} = \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle = \frac{4}{5} i - \frac{3}{5} j$$

(1) تأكد ان طوله

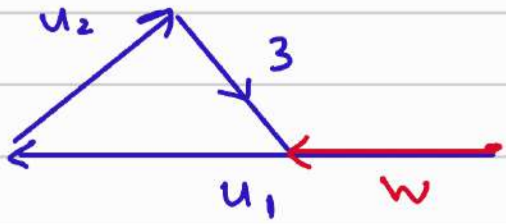
$$|\hat{u}| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2}$$

$$\sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1$$

Problem Set 11.2

Draw vector w

3. $\mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$



5. Figure 15 is a parallelogram. Express **w** in terms of **u** and **v**.

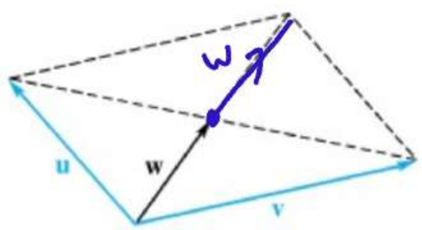
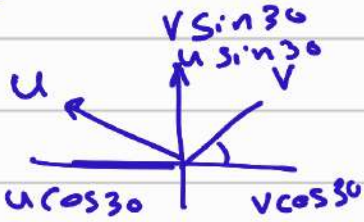


Figure 15

$$2w = u + v$$

$$w = \frac{1}{2}u + \frac{1}{2}v$$



7. In Figure 17, $\mathbf{w} = -(\mathbf{u} + \mathbf{v})$ and $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$. Find $\|\mathbf{w}\|$.

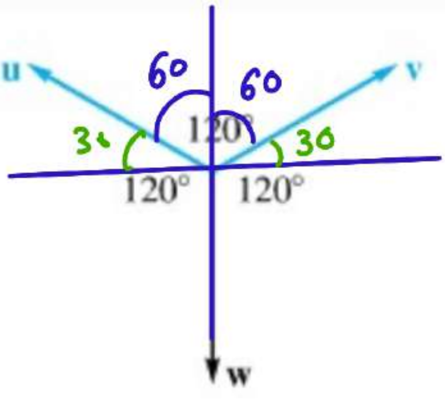


Figure 17

$$|W| = |V| \sin 30 + |U| \sin 30$$

$$|W| = 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)$$

$$|W| = \underline{1}$$

For the two-dimensional vectors **u** *and* **v** *in Problems 9–12, find the sum* **u** + **v**, *the difference* **u** − **v**, *and the magnitudes* $\|\mathbf{u}\|$ *and* $\|\mathbf{v}\|$.

9. $\mathbf{u} = \langle -1, 0 \rangle$, $\mathbf{v} = \langle 3, 4 \rangle$

10. $\mathbf{u} = \langle 0, 0 \rangle$, $\mathbf{v} = \langle -3, 4 \rangle$

11. $\mathbf{u} = \langle 12, 12 \rangle$, $\mathbf{v} = \langle -2, 2 \rangle$

12. $\mathbf{u} = \langle -0.2, 0.8 \rangle$, $\mathbf{v} = \langle -2.1, 1.3 \rangle$

9) $u + v = \langle -1, 0 \rangle + \langle 3, 4 \rangle = \langle 2, 4 \rangle$

$u - v = \langle -1, 0 \rangle - \langle 3, 4 \rangle = \langle -4, -4 \rangle$

$|u| = \sqrt{(-1)^2 + 0^2} = 1$

$|v| = \sqrt{3^2 + 4^2} = 5$

11) $u = \langle 12, 12 \rangle$ $\quad$ $v = \langle -2, 2 \rangle$

$u + v = \langle 12, 12 \rangle + \langle -2, 2 \rangle = \langle 10, 14 \rangle$

$u - v = \langle 12, 12 \rangle - \langle -2, 2 \rangle = \langle 14, 10 \rangle$

$|u| = \sqrt{12^2 + 12^2} = \sqrt{288}$

$|v| = \sqrt{2^2 + 2^2} = \sqrt{8}$

For the three-dimensional vectors **u** *and* **v** *in Problems 13–16, find the sum* $\mathbf{u} + \mathbf{v}$, *the difference* $\mathbf{u} - \mathbf{v}$, *and the magnitudes* $\|\mathbf{u}\|$ *and* $\|\mathbf{v}\|$.

13. $\mathbf{u} = \langle -1, 0, 0\rangle$, $\mathbf{v} = \langle 3, 4, 0\rangle$

14. $\mathbf{u} = \langle 0, 0, 0\rangle$, $\mathbf{v} = \langle -3, 3, 1\rangle$

15. $\mathbf{u} = \langle 1, 0, 1\rangle$, $\mathbf{v} = \langle -5, 0, 0\rangle$

13) $u+v = \langle -1,0,0\rangle + \langle 3,4,0\rangle = \langle 2,4,0\rangle$

$u-v = \langle -1,0,0\rangle - \langle 3,4,0\rangle = \langle -4,-4,0\rangle$

$|u| = \sqrt{(-1)^2+0^2+0^2} = 1$

$|v| = \sqrt{3^2+4^2} = 5$

15) $u = \langle 1,0,1\rangle \qquad v = \langle -5,0,0\rangle$

$u+v = \langle 1,0,1\rangle + \langle -5,0,0\rangle$

$= \langle -4,0,1\rangle$

$u-v = \langle 1,0,1\rangle - \langle -5,0,0\rangle$

$\langle 6,0,1\rangle$

$|u| = \sqrt{1^2+1^2} = \sqrt{2}$

$|v| = \sqrt{(-5)^2} = 5$

C **17.** In Figure 18, forces **u** and **v** each have magnitude 50 pounds. Find the magnitude and direction of the force **w** needed to counterbalance **u** and **v**.

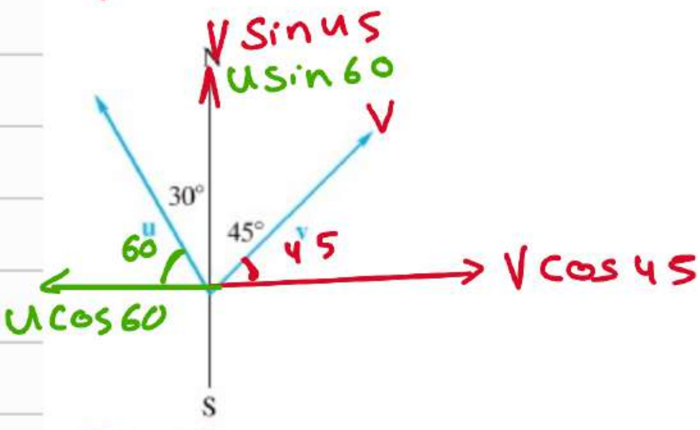


Figure 18

	sin	cos
30	$1/2$	$\sqrt{3}/2$
60	$\sqrt{3}/2$	$1/2$
45	$\sqrt{2}/2$	$\sqrt{2}/2$

$$R_x = V\cos 45 - u\cos 60$$

$$R_x = \frac{50\sqrt{2}}{2} - \frac{50}{2} = 25(\sqrt{2}-1)$$

$$R_y = V\sin 45 + u\sin 60$$

$$= \frac{50\sqrt{2}}{2} + \frac{50\sqrt{3}}{2} = 25(\sqrt{2}+\sqrt{3})$$

القوة موازنة

$$W_x = 25(1-\sqrt{2})$$

$$W_y = -25(\sqrt{2}+\sqrt{3})$$

$$W = \langle 25(1-\sqrt{2}), -25(\sqrt{2}+\sqrt{3}) \rangle$$

$$|w| = \sqrt{625(1-\sqrt{2})^2 + 625(\sqrt{2}+\sqrt{3})^2}$$

$$= 79.33$$

$$\theta = \tan^{-1}\left(\frac{-2\cancel{5}(\sqrt{2}+\sqrt{3})}{\cancel{25}(1-\sqrt{2})}\right) = 82.5$$

$$= 7.5$$

11.3
The Dot Product

الضرب النقطي (القياسي)

ضرب متجه بعدد

$u = \langle 2, 5 \rangle \qquad 3u = \langle 6, 15 \rangle$

Dot Product $\quad u \bullet v$

dot

ضرب متجه بمتجه آخر والناتج قياسي (رقم)

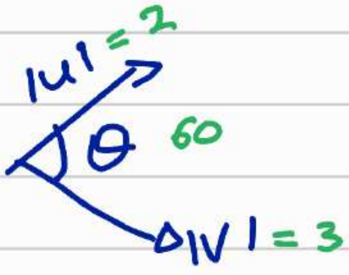
$$\boxed{u \bullet v = \langle u_1, u_2, u_3 \rangle \bullet \langle v_1, v_2, v_3 \rangle}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3 = \boxed{\text{رقم}}$$

$u = \langle 1, 2, 0 \rangle \qquad v = \langle 3, -1, 5 \rangle$

$$u \cdot v = 1(3) + 2(-1) + 0(5) = \boxed{1}$$

$$\boxed{u \cdot v = |u|\,|v|\cos\theta}$$



$$u \cdot v = 2(3)\cos 60 = 3$$

* عندما تكون زاوية 90 بين المتجهين $u \cdot v = 0$
* اكبر قيمة للضرب النقطي عندما تكون الزاوية صفر
$u \cdot v = |u||v|$
* عندما تكون الزاوية 180 (متعاكسين) $\theta = 180$
$u \cdot v = -|u||v|$

Theorem A **Properties of the Dot Product**

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors, and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
3. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v}$
4. $\mathbf{0} \cdot \mathbf{u} = 0$
5. $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$

خصائص الضرب النقطي

Theorem B

If θ is the smallest nonnegative angle between the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Theorem C **Perpendicularity Criterion**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if their dot product, $\mathbf{u} \cdot \mathbf{v}$, is 0.

متعامدين

Definition **Orthogonal**

Vectors that are perpendicular are said to be **orthogonal.**

المتجهات المتعامدة (90) $u \cdot v = 0$
تسمى othogonal

EXAMPLE 1 Let $\mathbf{u} = \langle 0, 1, 1\rangle$, $\mathbf{v} = \langle 2, -1, 1\rangle$, and $\mathbf{w} = \langle 6, -3, 3\rangle$. Compute each of the following if they are defined: (a) $\mathbf{u}\cdot\mathbf{v}$, (b) $\mathbf{v}\cdot\mathbf{u}$, (c) $\mathbf{v}\cdot\mathbf{w}$, (d) $\mathbf{u}\cdot\mathbf{u}$, and (e) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$.

a) $u \cdot v = \langle 0,1,1\rangle \cdot \langle 2,-1,1\rangle = 0 + (-1) + 1 = 0$

orthogonal $\theta = 90$

b) $v \cdot u \Rightarrow 0$ لان العمليه التبديلية صحيحه في الضرب

c) $v \cdot w = \langle 2,-1,1\rangle \cdot \langle 6,-3,3\rangle = 12 + 3 + 3 = 18$

d) $u \cdot u = \langle 0,1,1\rangle \langle 0,1,1\rangle = 0 + 1 + 1 = 2$

$|u| = \sqrt{2}$

e) $(\underline{u \cdot v}) \cdot w \Rightarrow$ no defined رقم $\cdot w$

رقم

[illegible]

حساب الزاويه بين المتجهين

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

$$\theta = \cos^{-1} \frac{u \cdot v}{|u||v|}$$

EXAMPLE 2 Find the angle between $\mathbf{u} = \langle 8, 6 \rangle$ and $\mathbf{v} = \langle 5, 12 \rangle$ (see Figure 2).

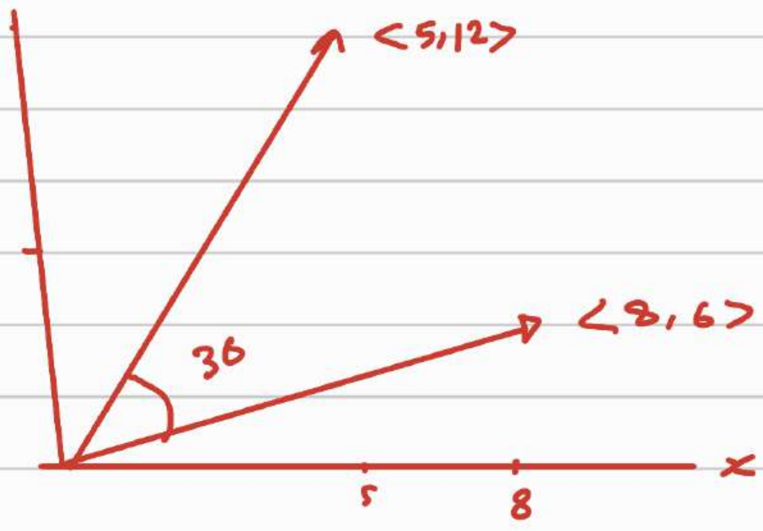
✓ $u \cdot v = \langle 8,6 \rangle \cdot \langle 5,12 \rangle = 40 + 72 = 112$

✓ $|u| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

✓ $|v| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|} = \frac{112}{10(13)} = \frac{112}{130}$$

$$\boxed{\cos\theta = \frac{112}{130}}$$



$\cos^{-1}\left(\frac{112}{130}\right) = 30.5$

EXAMPLE 3 Find the angles between each of the three pairs of vectors from Example 1. Which pairs are orthogonal?

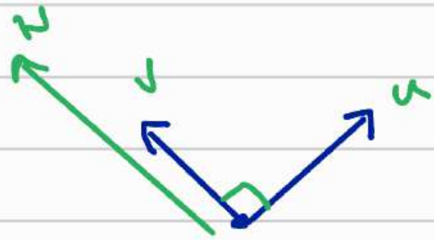
$u = \langle 0, 1, 1 \rangle$ $\quad |u| = \sqrt{2}$

$v = \langle 2, -1, 1 \rangle$ $\quad |v| = \sqrt{6}$

$w = \langle 6, -3, 3 \rangle$ $\quad |w| = \sqrt{54} = \sqrt{9 \cdot 6} = 3\sqrt{6}$

✓ θ_1 between u and v

$$\cos\theta_1 = \frac{u \cdot v}{|u||v|} = \frac{0}{\sqrt{2}\sqrt{6}} = 0$$



$$\theta_1 = 90$$

θ_2 between u w

$$\cos\theta_2 = \frac{u \cdot w}{|u|\,|w|} = \frac{0 - 3 + 3}{\sqrt{2} \cdot 3\sqrt{6}} = 0$$

$$\theta = 90$$

θ_3 betwee v and w

$$\cos\theta_3 = \frac{v \cdot w}{|v||w|} = \frac{2(6) + 3 + 3}{\sqrt{6} \cdot 3\sqrt{6}} = \frac{18}{18} = 1$$

$$\theta = \text{zero}$$

EXAMPLE 5 Find the measure of angle ABC if the points are $A(1, -2, 3)$, $B(2, 4, -6)$, and $C(5, -3, 2)$ (Figure 4).

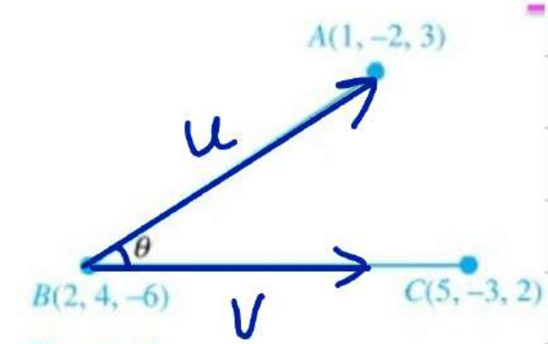


Figure 4

$u = \langle 1-2, -2-4, 3--6 \rangle$

$u = \langle -1, -6, 9 \rangle$

$v = \langle 5-2, -3-4, 2--6 \rangle$

$v = \langle 3, -7, 8 \rangle$

$u \cdot v = -1(3) + -6(-7) + 9(8)$
$= -3 + 42 + 72$
$= 111$

$|v| = \sqrt{9+49+64} = \sqrt{122}$

$|u| = \sqrt{1+36+81} = \sqrt{118}$

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

$$\cos\theta = \frac{111}{\sqrt{122}\sqrt{118}}$$

$\cos\theta = 0.925$

$\theta = \cos^{-1}(0.925) = 22.3$

Problem Set 11.3

1. Let $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$, and $\mathbf{c} = -5\mathbf{j}$. Find each of the following:

(a) $2\mathbf{a} - 4\mathbf{b}$

(b) $\mathbf{a} \cdot \mathbf{b}$

(c) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$

(d) $(-2\mathbf{a} + 3\mathbf{b}) \cdot 5\mathbf{c}$

(e) $\|\mathbf{a}\|\mathbf{c} \cdot \mathbf{a}$

(f) $\mathbf{b} \cdot \mathbf{b} - \|\mathbf{b}\|$

$\overset{a}{\langle -2,3\rangle}$ $\quad \overset{b}{\langle 2,-3\rangle}$ $\quad \overset{c}{\langle 0,-5\rangle}$

a) $2a - 4b = 2\langle -2,3\rangle - 4\langle 2,-3\rangle$

$= \langle -4, 6\rangle - \langle 8,-12\rangle$

$\langle -12, 18\rangle = -12i + 18j$

b) $a \cdot b = \langle -2,3\rangle \cdot \langle 2,-3\rangle$

$-2(2) + 3(-3) = -4 - 9 = -13$

c) $a \cdot (b+c) = \langle -2,3\rangle \cdot (\langle 2,-3\rangle + \langle 0,-5\rangle)$

$= \langle -2,3\rangle \cdot \langle 2,-8)$

$-2(2) + 3(-8) = -4 - 24 = -28$

d) $(-2a + 3b) \cdot 5c$

$(-2\langle -2,3\rangle + 3\langle 2,-3\rangle) \cdot (5\langle 0,-5\rangle)$

$(\langle 4,-6\rangle + \langle 6,-9\rangle) \cdot \langle 0,-25\rangle$

$\langle 10, -15\rangle \cdot \langle 0,-25\rangle$

$10(0) + (-15)(-25) = 375$

$$\overset{a}{\langle -2,3 \rangle} \qquad \overset{b}{\langle 2,-3 \rangle} \qquad \overset{c}{\langle 0,-5 \rangle}$$

e) $\|a\| \, c \cdot a$ $\qquad \|a\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

$\sqrt{13} \left(\langle 0,-5 \rangle \cdot \langle -2,3 \rangle \right)$

$\sqrt{13} \left(0 + (-15) = -15\sqrt{13} \right.$

$b \cdot b = \|b|^2$

f) $b \cdot b - \|b\| =$ $\qquad |b| = \sqrt{2^2 + -3^2} = \sqrt{13}$

$\langle 2,-3 \rangle \cdot \langle 2,-3 \rangle - \sqrt{13}$

$13 - \sqrt{13}$

2. Let $\mathbf{a} = \langle 3, -1 \rangle$, $\mathbf{b} = \langle 1, -1 \rangle$, and $\mathbf{c} = \langle 0, 5 \rangle$. Find each of the following:

$\|b\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\|c\| = \sqrt{0^2+5^2} = 5$

(a) $-4\mathbf{a} + 3\mathbf{b}$ (b) $\mathbf{b} \cdot \mathbf{c}$

(c) $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$ (d) $2\mathbf{c} \cdot (3\mathbf{a} + 4\mathbf{b})$

(e) $\|\mathbf{b}\|\mathbf{b} \cdot \mathbf{a}$ (f) $\|\mathbf{c}\|^2 - \mathbf{c} \cdot \mathbf{c}$ = zero

a) $-4a + 3b = -4\langle 3, -1 \rangle + 3\langle 1, -1 \rangle$

$= \langle -12, 4 \rangle + \langle 3, -3 \rangle$

$\langle -9, 1 \rangle$

b) $b \cdot c = \langle 1, -1 \rangle \cdot \langle 0, 5 \rangle$

$1(0) + (-1)(5) = -5$

c) $(a+b) \cdot c = \left(\langle 3, -1 \rangle + \langle 1, -1 \rangle\right) \cdot \langle 0, 5 \rangle$

$= \langle 4, -2 \rangle \cdot \langle 0, 5 \rangle = -10$

d) $2c \cdot (3a + 4b) =$

$(2\langle 0, 5 \rangle) \cdot (3\langle 3, -1 \rangle + 4\langle 1, -1 \rangle)$

$\langle 0, 10 \rangle \cdot (\langle 9, -3 \rangle + \langle 4, -4 \rangle)$

$\langle 0, 10 \rangle \cdot \langle 13, -7 \rangle = -70$

e) $\|b\| b \cdot a = \sqrt{2}\, \langle 3, -1 \rangle \cdot \langle 1, -1 \rangle$

$\sqrt{2}\,(3 + 1) = 4\sqrt{2}$

f) $\|c\|^2 - c \cdot c = 5^2 - 5^2 = 0$

3. Find the cosine of the angle between **a** and **b** and make a sketch.

(a) $\mathbf{a} = \langle 1, -3 \rangle, \mathbf{b} = \langle -1, 2 \rangle$ (b) $\mathbf{a} = \langle -1, -2 \rangle, \mathbf{b} = \langle 6, 0 \rangle$

(c) $\mathbf{a} = \langle 2, -1 \rangle, \mathbf{b} = \langle -2, -4 \rangle$

(d) $\mathbf{a} = \langle 4, -7 \rangle, \mathbf{b} = \langle -8, 10 \rangle$

$$\cos\theta = \frac{a \cdot b}{|a| \cdot |b|}$$

a)

$$\|a\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\|b\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\cos\theta = \frac{-7}{\sqrt{10}\sqrt{5}} = \frac{-7}{\sqrt{50}}$$

$$a \cdot b = -1 + -6 = -7$$

---

b)

$$\|a\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\|b\| = \sqrt{6^2 + 0^2} = 6$$

$$\cos\theta = \frac{-6}{6\sqrt{5}} = \frac{-1}{\sqrt{5}}$$

$$a \cdot b = -6 + = -6$$

---

c)

$$\|a\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\|b\| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$\cos\theta = \frac{0}{\sqrt{10}\sqrt{20}} = 0$$

$$a \cdot b = -4 + 4 = 0$$

---

d)

$$\|a\| = \sqrt{4^2 + 7^2} = \sqrt{65}$$

$$\|b\| = \sqrt{8^2 + 10^2} = \sqrt{164}$$

$$\cos\theta = \frac{-102}{\sqrt{65}\sqrt{164}}$$

$$a \cdot b = (-32) - 70 = -102$$

4. Find the angle between **a** and **b** and make a sketch.

(a) $\mathbf{a} = 12\mathbf{i}, \mathbf{b} = -5\mathbf{i}$

(b) $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}, \mathbf{b} = -8\mathbf{i} - 6\mathbf{j}$

(c) $\mathbf{a} = -\mathbf{i} + 3\mathbf{j}, \mathbf{b} = 2\mathbf{i} - 6\mathbf{j}$

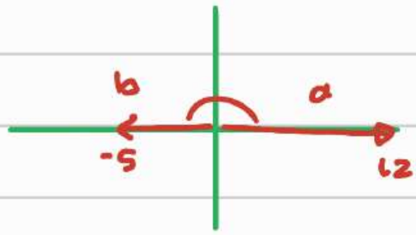
(d) $\mathbf{a} = \sqrt{3}\mathbf{i} + \mathbf{j}, \mathbf{b} = 3\mathbf{i} + \sqrt{3}\mathbf{j}$

$\cos\theta = \dfrac{a \cdot b}{\|a\|\|b\|}$

$\theta = \cos^{-1} \dfrac{a \cdot b}{|a||b|}$

(a)

a) $a = \langle 12, 0 \rangle \quad b = \langle -5, 0 \rangle$



$\|a\| = \sqrt{12^2} = 12$

$\|b\| = \sqrt{-5^2} = 5$

$a \cdot b = -60$

$\theta = \cos^{-1} \dfrac{-60}{60}$

$\theta = \cos^{-1}(-1)$

$\theta = 180$

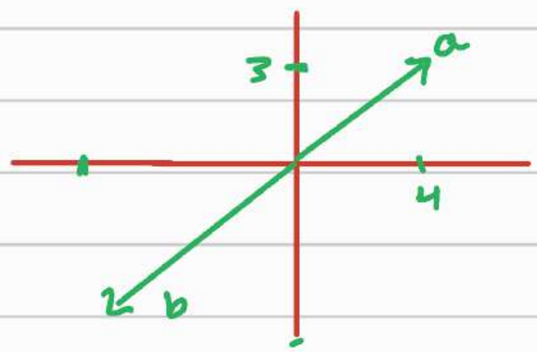
(b)

$a = \langle 4, 3 \rangle \quad b = \langle -8, -6 \rangle$

$\|a\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

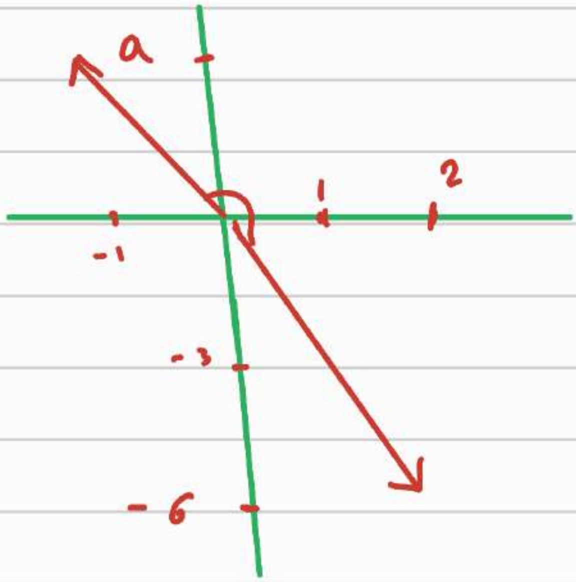
$\|b\| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

$a \cdot b = -32 - 18 = -50$



$\theta = \cos^{-1} \dfrac{-50}{5 \cdot 10} = \cos^{-1}(-1) \quad \theta = 180$

c) a) $\langle -1, 3 \rangle$ $b = \langle 2, -6 \rangle$



$||a|| = \sqrt{1^2+3^2} = \sqrt{10}$

$||b|| = \sqrt{2^2+6^2} = \sqrt{40}$

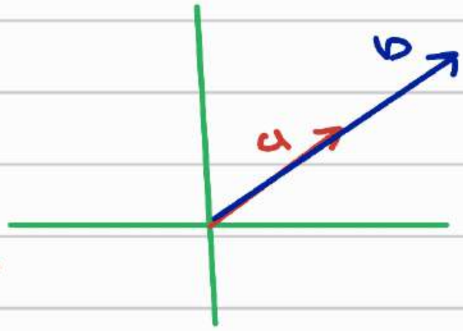
$a \cdot b = -2 - 18 = -20$

$\theta = \cos^{-1} \dfrac{-20}{\sqrt{10}\sqrt{40}} = \cos^{-1} \dfrac{-20}{20} = \cos^{-1}(-1)$

$\theta = 180$

---

d)



$||a|| = \sqrt{3^2+1^2} = \sqrt{10}$

$||b|| = \sqrt{3^2+\sqrt{3}^2} = \sqrt{12} = 3\sqrt{3}$

$a \cdot b = 3\sqrt{3} - \sqrt{3} = 2\sqrt{3}$

$\theta = \cos^{-1} \dfrac{2\sqrt{3}}{2\sqrt{3}} = \cos^{-1}(1)$

$\theta = 0$

6. Let $\mathbf{a} = \langle \sqrt{2}, \sqrt{2}, 0 \rangle$, $\mathbf{b} = \langle 1, -1, 1 \rangle$, and $\mathbf{c} = \langle -2, 2, 1 \rangle$. Find each of the following:

(a) $\mathbf{a} \cdot \mathbf{c}$

(b) $(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b}$

(c) $\mathbf{a}/\|\mathbf{a}\|$

(d) $(\mathbf{b} - \mathbf{c}) \cdot \mathbf{a}$

(e) $\dfrac{\mathbf{b} \cdot \mathbf{c}}{\|\mathbf{b}\|\|\mathbf{c}\|}$

(f) $\underbrace{\mathbf{a} \cdot \mathbf{a}}_{\|a\|^2} - \|\mathbf{a}\|^2 = 0$

a) $a.c = \langle \sqrt{2}, \sqrt{2}, 0 \rangle \cdot \langle -2, 2, 1 \rangle$

$= -2\sqrt{2} + 2\sqrt{2} + 0 = 0$

b) $(a - c) \cdot b$

$(\langle \sqrt{2}, \sqrt{2}, 0 \rangle - \langle -2, 2, 1 \rangle) \cdot \langle 1, -1, 1 \rangle$

$\langle \sqrt{2}+2, \sqrt{2}-2, -1 \rangle \cdot \langle 1, -1, 1 \rangle$

$\cancel{\sqrt{2}}+2 - \cancel{\sqrt{2}}+2 - 1 = 3$

c)

$\|a\| = \sqrt{\sqrt{2}^2 + \sqrt{2}^2 + 0^2} = \sqrt{4}$
$= 2$

$\dfrac{a}{\|a\|} = \dfrac{\langle \sqrt{2}, \sqrt{2}, 0 \rangle}{2}$

$= \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \rangle = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j$

d) $(b-c)\cdot a$

$$(\langle 1,-1,1\rangle - \langle -2,2,1\rangle)\cdot\langle\sqrt{2},\sqrt{2},0\rangle$$

$$\langle 3,-3,0\rangle\cdot\langle\sqrt{2},\sqrt{2},0\rangle$$

$$3\sqrt{2} + -3\sqrt{2} + 0 = 0$$

e) $$\frac{b\cdot c}{\|b\|\,\|c\|} = \frac{-2+(-2)+1}{\sqrt{3}\,\sqrt{9}} = -\frac{\sqrt{3}}{3}$$

f) $$a\cdot a - |a|^2 = 0$$

8. Let $\mathbf{a} = \langle \sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3 \rangle$, $\mathbf{b} = \langle 1, -1, 0 \rangle$, and $\mathbf{c} = \langle -2, -2, 1 \rangle$. Find the angle between each pair of vectors.

$$\|a\| = \sqrt{\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{3}}{3}\right)^2} = \sqrt{1} = 1$$

$\theta_{a,b}$

$$\|b\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$a \cdot b = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + 0 = 0$$

$$\theta = \cos^{-1} \frac{a \cdot b}{|a||b|} = \cos^{-1} \frac{0}{\sqrt{2}\sqrt{1}}$$

$$\theta = \cos^{-1}(0)$$

$$\theta_{a,b} = 90$$

$\theta_{a,c}$

$$\|a\| = 1$$

$$\|c\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$a \cdot c = -\frac{2\sqrt{3}}{3} - \frac{2\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = -\frac{\cancel{3}\sqrt{3}}{\cancel{3}} = -\sqrt{3}$$

$$\theta = \cos^{-1} \frac{-\sqrt{3}}{3} = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = 125$$

12. Show that the vectors $\mathbf{a} = \langle 1, 1, 1 \rangle$, $\mathbf{b} = \langle 1, -1, 0 \rangle$, and $\mathbf{c} = \langle -1, -1, 2 \rangle$ are mutually orthogonal, that is, each pair of vectors is orthogonal.

$$a \cdot b = 1 + (-1) + 0 = 0$$

$$b \cdot c = -1 + 1 + 0 = 0$$

$$a \cdot c = -1 + (-1) + 2 = 0$$

All orthogonal

b **16.** Find all vectors perpendicular to both $\langle 1, -2, -3 \rangle$ (a) and $\langle -3, 2, 0 \rangle$.

العمودي على كل من المتجهين $= \langle x, y, z \rangle = n$

$$n \cdot a = 0 \Rightarrow x - 2y - 3z = 0$$

$$n \cdot b = 0 \Rightarrow -3x + 2y = 0$$

نستبدلها بالاولى

$$y = \frac{3x}{2}$$

$$x - \cancel{2}\left(\frac{3x}{\cancel{2}}\right) - 3z = 0$$

$$z = -\frac{2x}{3}$$

$$\left\langle x, \frac{3x}{2}, -\frac{2}{3}x \right\rangle$$

21. For what numbers c and d are $\mathbf{u} = c\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + d\mathbf{k}$ orthogonal?

$u \cdot v = 0$

$\langle c, 1, 1 \rangle \cdot \langle 0, 2, d \rangle$

$\cancel{c(0)} + 2 + d = 0$

$\boxed{c = \text{any number}}$ $\boxed{d = -2}$

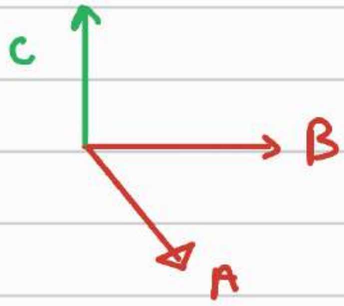
11.4 The Cross Product

$\vec{A} \times \vec{B}$

الضرب الاتجاهي (التقاطعي)

ضرب متجهين والناتج هو متجه جديد عمودي على كل من المتجهين

$A \times B = C$



$$\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix}$$

محددة $ad - bc$

$15 - 2 = 13$

Find the cross product $\vec{a} \times \vec{b}$

$a = \langle 3, 4, 5 \rangle \qquad b = \langle 7, 8, 9 \rangle$

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & 4 & 5 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 5 \\ 8 & 9 \end{vmatrix} i - \begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix} j + \begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix} k$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$(4(9) - 5(8))\, i - (3(9) - 5(7))\, j + (3(8) - 7(4))\, k$$

$$-4i - (-8)j - 4k = -4i + 8j - 4k$$

$\langle -4, 8, -4 \rangle$

Theorem A

Let **u** and **v** be vectors in three-space and θ be the angle between them. Then

1. $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0 = \mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})$, that is, $\mathbf{u}\times\mathbf{v}$ is perpendicular to both **u** and **v**;
2. **u**, **v**, and $\mathbf{u}\times\mathbf{v}$ form a right-handed triple;
3. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$.

عمودي

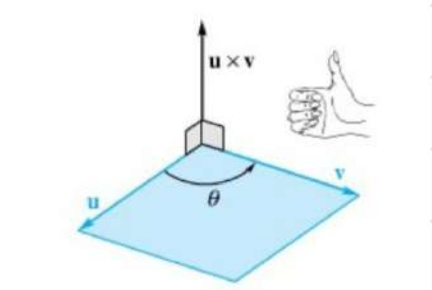
① $u \times v = -(v \times u)$ anticommutative

② المتجه الناتج من الضرب الاتجاهي يكون عمودي على المتجهين

$(u\times v)\cdot u = 0$

$(u\times v)\cdot v = 0$

③ لتحديد اتجاه الضرب للعناصر نستخدم اليد اليمنى



$$\boxed{|u \times v| = |u|\,|v| \sin\theta}$$

$|u| = 4$ $\quad |v| = 5$ $\quad \theta = 30^\circ$

$$|u \times v| = 4(5)\sin 30 = 10$$

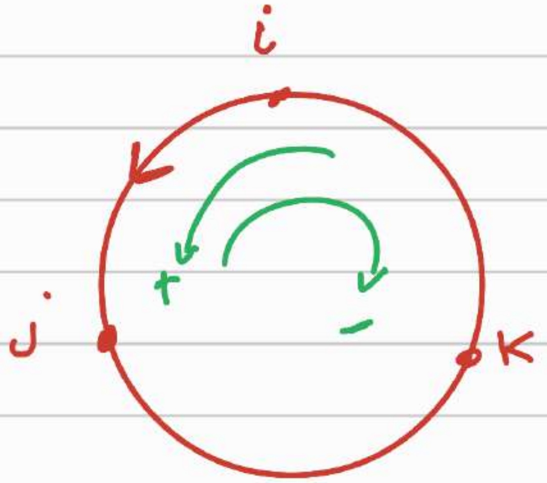
اذا كان المتجهين بنفس الاتجاه $\theta = 0, 180$

فان الضرب الاتجاهي = صفر (Parallel)

اذا كان المتجهين متعامدين (Perpendicular)

$\theta = 90$

$$u \times v = |u|\,|v|$$



$i \times j = k$ $\quad j \times i = -k$

$j \times k = i$ $\quad k \times j = -i$

$k \times i = j$ $\quad i \times k = -j$

EXAMPLE 1 Let $\mathbf{u} = \langle 1, -2, -1\rangle$ and $\mathbf{v} = \langle -2, 4, 1\rangle$. Calculate $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ using the determinant definition.

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & -1 \\ 4 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} k$$

$$(-2(1) - (-1)(4))i - (1 - (-1)(-2))j + (4 - (-2)(-2))k$$

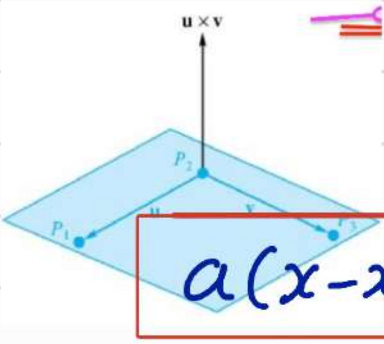
$$(-2+4)i - (1-2)j + (4-4)k$$

$$u \times v = 2i + j + 0k$$

$$v \times u = -2i - j + 0k$$

لكتابة معادله المستوى نحتاج العمودي
عليه وايضا نقطه ع المستوى

النقطه (x_1, y_1, z_1)



العمودي $\vec{n} = ai + bj + ck$

$$\boxed{a(x-x_1) + b(y-y_1) + c(z-z_1) = 0}$$

EXAMPLE 2 Find the equation of the plane (Figure 2) through the three points $P_1(1, -2, 3)$, $P_2(4, 1, -2)$, and $P_3(-2, -3, 0)$.

اولا نحسب العمودي $u \times v$

$u = \overrightarrow{P_2 P_1}$

$u = \langle 1-4, -2-1, 3--2 \rangle = \langle -3, -3, 5 \rangle$

$v = \overrightarrow{P_2 P_3}$

$v = \langle -2-4, -3-1, 0--2 \rangle = \langle -6, -4, 2 \rangle$

$$u \times v = \begin{vmatrix} i & j & k \\ -3 & -3 & 5 \\ -6 & -4 & 2 \end{vmatrix} = \begin{vmatrix} -3 & 5 \\ -4 & 2 \end{vmatrix} i - \begin{vmatrix} -3 & 5 \\ -6 & 2 \end{vmatrix} j + \begin{vmatrix} -3 & -3 \\ -6 & -4 \end{vmatrix} k$$

$$= (-6+20)i - (-6+30)j + (12-18)k$$

العمودي $= u \times v = 14i - 24j - 6k$

نعوض في معادله السطح

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$
$$14(x-4) - 24(y-1) - 6(z+2) = 0$$

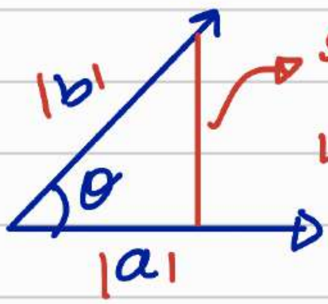
$$14x - 56 - 24y + 24 - 6z - 12 = 0$$

$$14x - 24y - 6z = +56 - 24 + 12$$

$$\boxed{14x - 24y - 6z = 44}$$

EXAMPLE 3 Show that the area of a parallelogram with **a** and **b** as adjacent sides is $\|\mathbf{a} \times \mathbf{b}\|$.

متوازي الاضلاع



$\sin\theta = \dfrac{L}{|b|}$

$L = |b| \sin\theta$

المساحة = القاعدة x الارتفاع

$A = |a|\,|b| \sin\theta$

$$\text{Area} = \| a \times b \|$$

$$= |a|\,|b| \sin\theta$$

Theorem C

If **u**, **v**, and **w** are vectors in three-space and k is a scalar, then

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (anticommutative law);
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ (left distributive law);
3. $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$;
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$, $\mathbf{u} \times \mathbf{u} = \mathbf{0}$;
5. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$;
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

EXAMPLE 5 Calculate $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

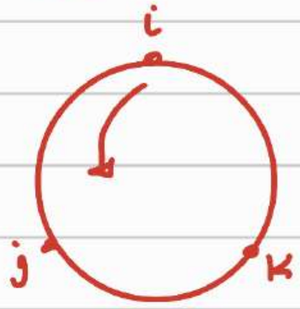
$$U \times V = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 4 & 2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix} i - \begin{vmatrix} 3 & 1 \\ 4 & -3 \end{vmatrix} j + \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix} k$$

$$(6-2)i - (-9-4)j + (6+8)k$$

$$U \times V = 4i + 13j + 14k$$

طريقة اخرى



$$U \times V = (3i - 2j + k) \times (4i + 2j - 3k)$$

$$= 12\cancel{(i \times i)} + 6\overset{k}{(i \times j)} - 9\overset{-j}{(i \times k)} - 8\overset{-k}{(j \times i)} - 4\cancel{(j \times j)}$$

$$+ 6\overset{i}{(j \times k)} + 4\overset{j}{(k \times i)} + 2\overset{-i}{(k \times j)} - 3\cancel{(k \times k)}$$

$$6k + 9j + 8k + 6i + 4j - 2i$$

$$4i + 13j + 14k$$

Problem Set 11.4

$\langle 7,3,-4\rangle$ **1.** Let $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, and $\mathbf{c} = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. Find each of the following:

$\langle -3,2,-2\rangle$ $\langle -1,2,-4\rangle$

(a) $\mathbf{a} \times \mathbf{b}$ (b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$

(c) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ (d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

a) $$a \times b = \begin{vmatrix} i & j & k \\ -3 & 2 & -2 \\ -1 & 2 & -4 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -2 \\ 2 & -4 \end{vmatrix} i - \begin{vmatrix} -3 & -2 \\ -1 & -4 \end{vmatrix} j + \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} k$$

$$(-8+4)i - (12-2)j + (-6+2)k$$

$$-4i - 10j - 4k$$

b) $$(b+c) = \langle -1,2,-4\rangle + \langle 7,3,-4\rangle$$
$$b+c = \langle 6,5,-8\rangle$$

$$a \times (b+c) = \begin{vmatrix} i & j & k \\ -3 & 2 & -2 \\ 6 & 5 & -8 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 5 & -8 \end{vmatrix} i - \begin{vmatrix} -3 & -2 \\ 6 & -8 \end{vmatrix} j + \begin{vmatrix} -3 & 2 \\ 6 & 5 \end{vmatrix} k$$

$$-6i - 36j + -27k$$

c) $b+c = \langle 6, 5, -8 \rangle$

$a \cdot (b+c) = \langle -3, 2, -2 \rangle \cdot \langle 6, 5, -8 \rangle$

$-18 + 10 + 16 = 8$

d) $b \times c = \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 7 & 3 & -4 \end{vmatrix}$

$= \begin{vmatrix} 2 & -4 \\ 3 & -4 \end{vmatrix} i - \begin{vmatrix} -1 & -4 \\ 7 & -4 \end{vmatrix} j + \begin{vmatrix} -1 & -2 \\ 7 & 3 \end{vmatrix} k$

$b \times c = 4i - 32j - 17k$

$a \times (b \times c) = \begin{vmatrix} i & j & k \\ -3 & 2 & -2 \\ 4 & -32 & -17 \end{vmatrix}$

$= \begin{vmatrix} 2 & -2 \\ -32 & -17 \end{vmatrix} i - \begin{vmatrix} -3 & -2 \\ 4 & -17 \end{vmatrix} j + \begin{vmatrix} -3 & 2 \\ 4 & -32 \end{vmatrix} k$

$(-34 - 64)i - (51 + 8)j + (96 - 8)k$

$-98i - 59j + 88k$

3. Find all vectors perpendicular to both of the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.

ناتج الضرب الاتجاهي هو عمودي على متجهين

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -2 & 2 & -4 \end{vmatrix}$$

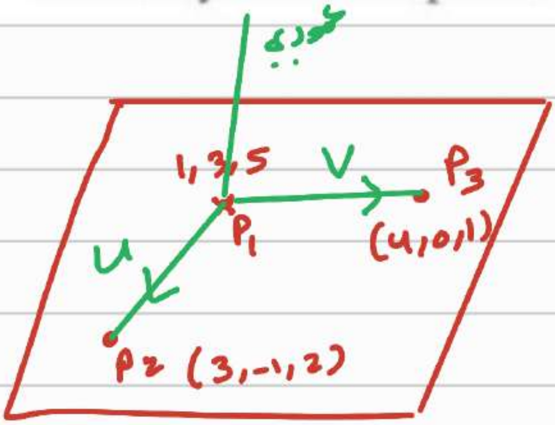
$$\begin{vmatrix} 2 & 3 \\ 2 & -4 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} k$$

$$(-8-6)i - (-4+6)j + (2+4)k$$

$$a \times b = -14i - 2j + 6k$$

$$b \times a = 14i + 2j - 6k$$

5. Find the unit vectors perpendicular to the plane determined by the three points $(1, 3, 5)$, $(3, -1, 2)$, and $(4, 0, 1)$.



$u : P_1 \longrightarrow P_2$

$u = \langle 3-1, -1-3, 2-5 \rangle$

$u = \langle 2, -4, 3 \rangle$

$V = \langle 4-1, 0-3, 1-5 \rangle$

$\langle 3, -3, -4 \rangle$

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & -4 & -3 \\ 3 & -3 & -4 \end{vmatrix} = \begin{vmatrix} -4 & -3 \\ -3 & -4 \end{vmatrix} i - \begin{vmatrix} 2 & -3 \\ 3 & -4 \end{vmatrix} j + \begin{vmatrix} 2 & -4 \\ 3 & -3 \end{vmatrix} k$$

$n =$ العمودي $= 7i - j + 6k = \langle 7, -1, 6 \rangle$

$$u = \frac{n}{\|n\|} = \frac{\langle 7, -1, 6 \rangle}{\sqrt{7^2 + 1^2 + 6^2}}$$

$$= \frac{\langle 7, -1, 6 \rangle}{\sqrt{86}}$$

$$\hat{u} = \left\langle \frac{7}{\sqrt{86}}, \frac{-1}{\sqrt{86}}, \frac{6}{\sqrt{86}} \right\rangle$$

8. Find the area of the parallelogram with $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ as the adjacent sides.

مساحه متوازي الاضلاع = || الضرب الاتجاهي |

$$\text{Area} = \| a \times b \|$$

$$a \times b = \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ -1 & 1 & -4 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ -1 & -4 \end{vmatrix} j + \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} k$$

$$a \times b = -7i + 9j + 4k$$

$$\| a \times b \| = \sqrt{-7^2 + 9^2 + 4^2} = \sqrt{146}$$

In Problems 11–14, find the equation of the plane through the given points.

11. $(1,3,2)$, $(0,3,0)$, and $(2,4,3)$

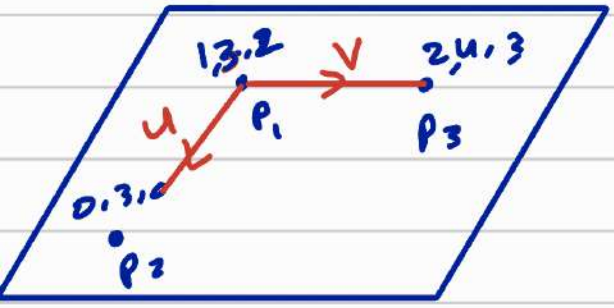
① تحديد متجهين في المستوى u v

② حساب المتجه العمودي على المستوى $n = u \times v$

③ نطبق معادله المستوى

$n = ai + bj + ck$ (x_1, y_1, z_1)

$$\boxed{a(x-x_1) + b(y-y_1) + c(z-z_1)}$$



$u = P_1 \rightarrow P_2$

$u = \langle 0-1, 3-3, 0-2 \rangle = \langle -1, 0, -2 \rangle$

$v = P_1 \rightarrow P_3$

$\langle 2-1, 4-3, 3-2 \rangle = \langle 1, 1, 1 \rangle$

$$u \times v = \begin{vmatrix} i & j & k \\ -1 & 0 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

$n = 2i - j - k$ عمودي $(1,2,3)$

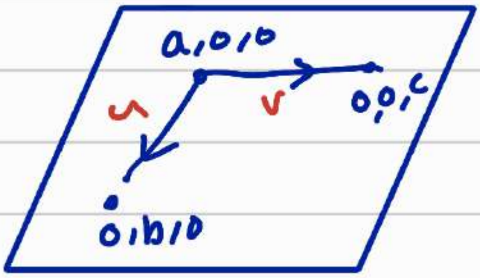
$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

$2(x-1) - 1(y-2) - 1(z-3) = 0$

$2x - 2 - y + 2 - z + 3 = 0$

$2x - y - z = -3$

14. $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, (None of a, b, and c is zero.)



$u = \langle -a, b, 0 \rangle$

$v = \langle -a, 0, c \rangle$

$$u \times v = \begin{vmatrix} i & j & k \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$$

$$u \times v = (bc)\, i + ac\, j + (ab)\, k$$

$(a, 0, 0)$

$$bc(x - a) + ac(y - 0) + ab(z - 0) = 0$$

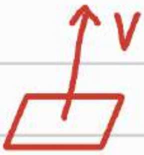
$$bcx - bca + acy + abz = 0$$

$$bcx + acy + abz = bca$$

$x_1 \quad y_1 \quad z_1$

18. Find the equation of the plane through $(2, -1, 4)$ that is perpendicular to both the planes $x + y + z = 2$ and $x - y - z = 4$.

$u = \langle 1, 1, 1 \rangle$



$v = \langle 1, -1, -1 \rangle$

لحساب العمودي على كل من المستويين

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 0i + 2j - 2k$$

$$\cancel{a(x - x_1)} + b(y - y_1) + c(z - z_1) = 0$$

$$2(y + 1) - 2(z - 4) = 0$$

$$2y + 2 - 2z + 8 = 0$$

$$\boxed{2y - 2z = -10}$$

23. Find the volume of the parallelepiped with edges $\langle 2, 3, 4\rangle$, $\langle 0, 4, -1\rangle$, and $\langle 5, 1, 3\rangle$ (see Example 4).

a b c

قانون حجم

قاعدة

$$V = |a \cdot (b \times c)|$$

مسقط رأسي

$$b \times c = \begin{vmatrix} i & j & k \\ 0 & 4 & -1 \\ 5 & 1 & 3 \end{vmatrix} = 13i - 5j - 20k$$

$$a \cdot (b \times c) = \langle 2, 3, 4\rangle \cdot \langle 13, -5, -20\rangle$$

$$26 - 15 - 80 = -69$$