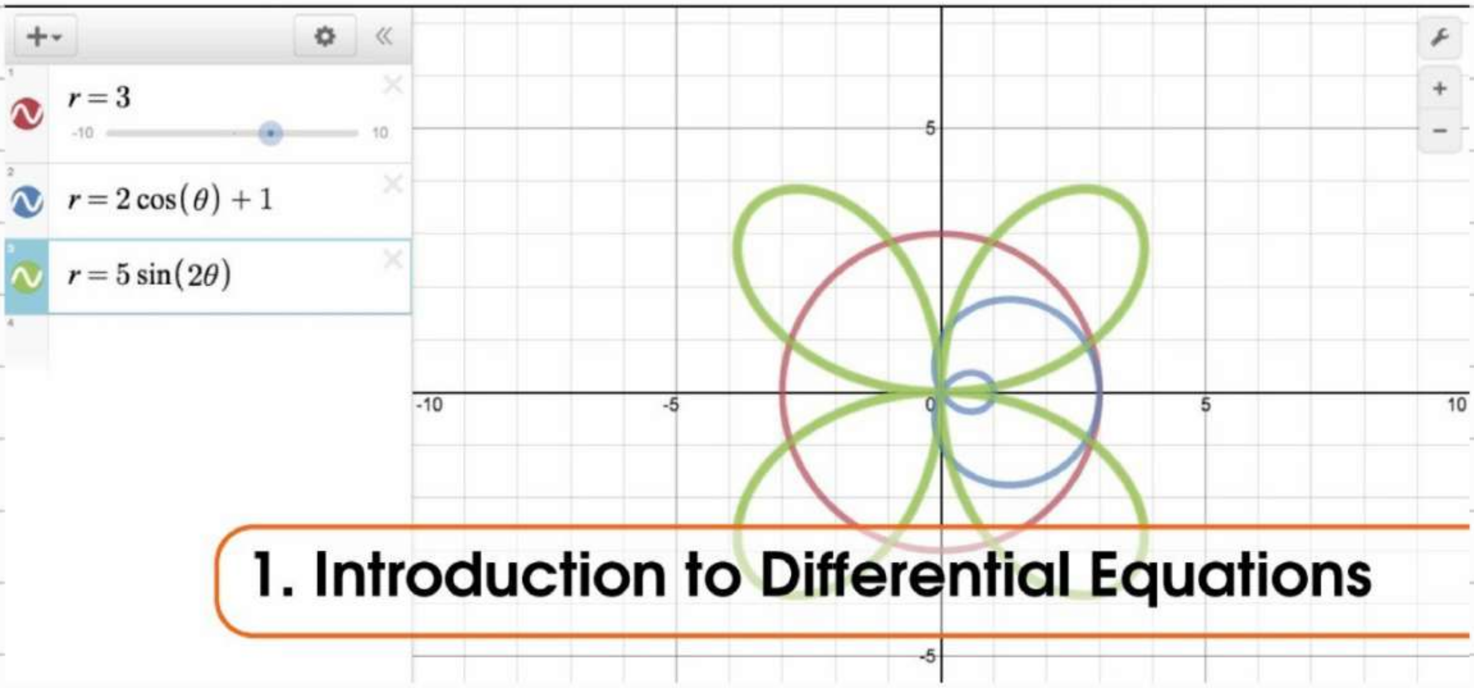

1. Introduction to Differential Equations

$x + 4 = 5$ $\qquad$ $x^2 + x - 6 = 0$

معادلات تفاضلية هي المعادلات التي تحتوي على مشتقات (مشتقة أولى $\frac{dy}{dx}$ ، ثانية $\frac{d^2y}{dx^2}$ ، ثالثة ...) وتحتوي على متغير مستقل واحد أو أكثر ومتغير تابع واحد أو أكثر

مشتقة أولى: $\frac{dy}{dx}$ $\qquad$ $\frac{dx}{dt}$ $\qquad$ y'

معادلة تفاضلية: $\frac{dy}{dx} + x - 6 = 0$

مشتقة ثانية: $\frac{d^2y}{dx^2}$ $\qquad$ y'' $\qquad$ $\frac{d^2u}{dx^2}$

$\frac{dy}{dx}$

- ind — x متغير مستقل
- dep — y متغير تابع

$\frac{du}{dv}$

- dep — u متغير تابع
- ind — v متغير مستقل

1.1 Definitions

Definition 1.1.1 An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).

Types of Differential Equations: There are two types of differential equations:

1. Ordinary differential equations (ODE)
2. Partial differential equations (PDE)

1. Ordinary Differential Equation: If an equation contains only ordinary derivatives of one or more dependent variables with respect to a $\boxed{\text{single independent variable}}$ it is said to be an ordinary differential equation (ODE)

Example: (1) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0$, (2) $\left(\frac{dy}{dx}\right)^3 - \left(\frac{dy}{dx}\right)^2 + 7y = \cos x$ → ind ⇒ x
dep ⇒ y

The general form of an ordinary differential equation is

تحتوي على متغير مستقل (x)
متغير تابع (y)

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots \frac{d^ny}{dx^n}\right) = 0$$

∂

2. Partial Differential Equation: An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a partial differential equation (PDE).

ind ⇒ x, y
dep ⇒ u

ind ⇒ x, t
dep ⇒ y

Example: (1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, (2) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$.

$\frac{d}{dx}$ (ODE) معادله تفاضليه اعتياديه

تحتوي على متغير مستقل واحد فقط ويمكن
ان تحتوي على متغير تابع واحد او اكثر

ODE ✓ $\frac{dy}{dx} + 3x + 5 = 0$ — مستقل x — تابع y

ODE ✓ $\frac{dy}{dx} + \frac{du}{dx} + 5 = 0$ — مستقل x — تابع y u

$\frac{\partial}{\partial x}$ PDE معادله تفاضليه جزئيه

يمكن ان تحتوي على متغير مستقل واحد او اكثر
ومتغير تابع واحد او اكثر

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 5$ — 2 متغير مستقل x y
متغير تابع واحد u

order : (رتبة) اعلى مشتقة في المعادلة
degree : (درجة) اس اعلى مشتقة في المعادلة

$\frac{dy}{dx} \Rightarrow$ مشتقة اولى

$\left(\frac{dy}{dx}\right)^2 =$ مشتقة اولى اس 2

$\frac{d^2y}{dx^2} \Rightarrow$ مشتقة ثانية

$\left(\frac{d^3y}{dx^3}\right)^2 =$ مشتقة ثالثة اس 2

Definition 1.1.2 The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation.

Example: $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$, Order$\rightarrow$ 2.

مشتقة ثانية

مشتقة اولى

Order = 2
degree = 1

Definition 1.1.3 The degree of a DE is the highest degree of the highest derivative in the DE.

Example: $(1-x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 3x = 0$, Order$\rightarrow$ 2, Degree $\rightarrow$ 1.

مشتقة ثانية

مشتقة اولى

Order = 2
degree = 1

Differential Equation	DV	IV	Type	Order	Degree	Linearity
$\frac{dy}{dx} = 5x$	y	x	ODE	1	1	Linear
$\frac{d^2y}{dx^2} + y = 0$	y	x	ODE	2	1	Linear
$\left(\frac{dy}{dx}\right)^2 + y = 0$	y	x	ODE	1	2	Non linear
$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$	u	x,t	PDE	1	1	linear
$y\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$	y	x	ODE	2	1	Nonlinear
$\frac{d^3y}{dx^3} = x$	y	x	ODE	3	1	Linear
$\left(\frac{d^2y}{dx^2}\right)^3 + y = 0$	y	x	ODE	2	3	Nonlinear

1.2 Linear and Nonlinear DEs

معادلات تفاضلية خطية
وغير خطية

Definition 1.2.1 A DE is said to be linear if
- the dependent variable and all its derivatives are of the first degree,
- each coefficient depends only on the independent variable.

Otherwise it is nonlinear.

$y \quad y' \quad y'' \quad y''' \quad y^{(4)}$ — مشتقة

y^4 — اس

Example 1.1 1. $\frac{d^3y}{dx^3} + y^2 = 0 \rightarrow$ nonlinear (power not 1)

2. $\frac{d^2y}{dx^2} + \sin y = 0 \rightarrow$ nonlinear (nonlinear function of y)

3. $(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 6y = e^x \rightarrow$ linear

4.

S.NO	Differential Equation	Order	Linear (or) Nonlinear	degree
1	$x\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^4 + y = 0$	3	Non linear	1
2	$t^5y^{(4)} - t^3y'' + 6y = 0$	4	Linear	1
3	$\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$	2	Nonlinear	1
4	$(\sin\theta)y''' - (\cos\theta)y'' = 2$	3	linear	1
5	$\frac{d^2u}{dr^2} + \frac{du}{dr} + u = \cos(r+u)$	2	Non linear	1
6	$(1-x)y'' - 4xy' + 5y = \cos x$	2	Linear	1

حتى تكون المعادلة التفاضلية خطية يجب ان يتحقق الركنان

① المتغير التابع (y) ومشتقاته من الدرجة الاولى (بدون اسس)

$y \,,\, \frac{dy}{dx} \,,\, \frac{d^2y}{dx^2} \,,\, \frac{d^3y}{dx^3} \,\ldots$

$\left(\frac{dy}{dx}\right)^3$ ممنوع

② معاملات المتغير التابع (y) ومشتقاته تكون اما دوال فيها x او ثوابت وممنوع (y)

ممنوع y → $\_\_\, y \,,\, \_\_\, \frac{dy}{dx} \,,\, \_\_\, \frac{d^2y}{dx^2}$

$x^2\frac{dy}{dx} + 3y = \sin x$ → linear

■ **Example 1.2** Determine whether the given first-order differential equation is linear (or) nonlinear in the indicated dependent variable.

$$(a)\ (y^2-1)dx + xdy = 0 \ \ in\ y; in\ x \quad (b)\ \ udv + (v + uv - ue^u)du = 0 \ \ in\ v;\ \ in\ u$$

■

a) $(y^2-1)\dfrac{\cancel{dx}}{\cancel{dx}} + x\dfrac{dy}{dx} = 0$

على اعتبار ان المتغير التابع DV هو y

$\dfrac{dy}{dx}$

نقسم المعادله على dx

$(y^2-1) + x\dfrac{dy}{dx} = 0$

$x\dfrac{dy}{dx} + y^{2} = 1$ w.r.t

هل هذه المعادله خطيه

Non linear (y power is not 1)

$(y^2-1)\dfrac{dx}{dy} + x\dfrac{\cancel{dy}}{\cancel{dy}} = 0$

على اعتبار ان x هي المتغير التابع

$\dfrac{dx}{dy}$

نقسم كل المعادله على dy

$(y^2-1)\dfrac{dx}{dy} + x = 0$

هل هذه المعادله خطيه

Linear DE in x

b) $u\frac{dv}{du} + (v+uv-ue^u)\cancel{\frac{du}{du}} = 0$

نقسم كل المعادله على du

على اعتبار ان
v هي المتغير التابع
و u هي المتغير المستقل
$\frac{dv}{du}$

$$u\frac{dv}{du} + (v + \underline{uv} - ue^u) = 0$$

$$u\frac{dv}{du} + (1+u)v = ue^u$$

هل هذه المعادله خطيه

linear in v

$$u\cancel{\frac{dv}{dv}} + (v+uv-ue^u)\frac{du}{dv} = 0$$

على اعتبار ان u
هي المتغير التابع
و v هي المتغير المستقل
$\frac{du}{dv}$

$$(v+uv-ue^u)\frac{du}{dv} + u = 0$$

هل هذه المعادله خطيه

Non linear in u

1.3 Solution of DE & Verification of solution

$\frac{d^2y}{dx^2} + 4y = 5e^x$ هو حلا للمعادله $y = e^x$ اثبت ان

نعوض الحل في المعادله واذا كان الطرفين متساويين اذا هو حل صحيح

$$\frac{d^2y}{dx^2} + 4y = 5e^x$$

مشتقه ثانيه

$y = e^x \qquad \frac{dy}{dx} = e^x \qquad \frac{d^2y}{dx^2} = e^x$

$$e^x + 4e^x \stackrel{?}{=} 5e^x$$

$$5e^x = 5e^x$$

$$LHS = RHS$$

✓

■ **Example 1.3** (i) Verify that $y_p = -\frac{1}{3}e^x$ is a particular solution of $y'' - 4y = e^x$
(ii) Verify that indicated function $y = e^{3x}\cos 2x$ is a solution of $y'' - 6y' + 13y = 0$
(iii) Verify that $y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{3}e^x$ is the general solution of $y'' - 4y = e^x$ ■

(i) $y_p = -\frac{1}{3}e^x$ هل هو حل $y'' - 4y = e^x$

$\boxed{y_p = -\frac{1}{3}e^x}$ $\qquad y_p' = -\frac{1}{3}e^x \qquad$ $\boxed{y_p'' = -\frac{1}{3}e^x}$

نعوض في المعادلة:

$$-\frac{1}{3}e^x - 4\left(-\frac{1}{3}e^x\right) \stackrel{?}{=} e^x$$

$$-\frac{1}{3}e^x + \frac{4}{3}e^x = e^x$$

$$\cancel{\frac{3}{3}}e^x = e^x$$

$$e^x = \quad LHS = RHS$$

Yes y_p is a solution

(ii) $\quad y = e^{3x}\cos 2x \qquad\qquad y'' - 6y' + 13y = 0$

$$y = e^{3x}\cos 2x \qquad\qquad y' = -2e^{3x}\sin 2x + 3e^{3x}\cos 2x$$

$$y'' = -4e^{3x}\cos 2x - 6e^{3x}\sin 2x - 6e^{3x}\sin 2x + 9e^{3x}\cos 2x$$

$$y'' = 5e^{3x}\cos 2x - 12e^{x}\sin 2x$$

$$y'' - 6y + 13y = 0$$

$$5e^{3x}\cos 2x \cancel{-12e^{3x}\sin 2x} \cancel{+12e^{3x}\sin 2x} - 18e^{3x}\cos 2x + 13e^{3x}\cos 2x$$

$$-13e^{3x}\cancel{\cos 2x + 13e^{3x}}\cos 2x = 0 \qquad\qquad 0 = 0 \ ✓$$

$$LHS = RHS$$

y	y'
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\sin 2x$	$2\cos 2x$
$\cos 3x$	$-3\sin 3x$
e^{x}	e^{x}
e^{3x}	$3e^{3x}$

y	y'
$f+g$	$f'+g'$
$3x+\sin x$	$3+\cos x$
$f\cdot g$	$fg'+f'g$
$x\sin x$	$x\cos x+\sin x$
$e^{3x}\cos x$	$-e^{3x}\sin x+3e^{3x}\cos x$

(iii) $$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3}e^{x}$$

$$y'' - 4y = e^{x}$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3}e^{x}$$

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{3}e^{x}$$

بعوض في المعادل

$$y'' = 4c_1 e^{2x} + 4c_2 e^{-2x} - \frac{1}{3}e^{x}$$

$$y'' - 4y = e^{x}$$

$$4c_1 e^{2x} + 4c_2 e^{-2x} - \frac{1}{2}e^{x} - 4\left(c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3}e^{x}\right) \stackrel{?}{=} e^{x}$$

$$\cancel{4c_1 e^{2x}} + \cancel{4c_2 e^{-2x}} - \frac{1}{3}e^{x} - \cancel{4c_1 e^{2x}} - \cancel{4c_2 e^{-2x}} + \frac{4}{3}e^{x} \stackrel{?}{=} e^{x}$$

$$\cancel{\frac{3}{3}}e^{x} \stackrel{?}{=} e^{x}$$

$$LHS = RHS \checkmark$$

1.4 Initial-Value Problem

IVP

IV خلال حل المعادله التفاضليه بعض شرط اوليه
نستخدم لحساب الثابت C

بعد حل المعادله التفاضليه كان الحل هو

$$y = ce^x$$

احسب قيمه الثابت C اذا كان

$y(0) = 5$

$x = 0$ $y = 5$

$$y = ce^x$$
$$5 = c\cancel{e^0}$$

$\boxed{C = 5}$

Example 1.4 If $y = \frac{1}{x^2+c}$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

$$(a)\ \ y(2) = \frac{1}{3}, \quad (b)\ \ y(-2) = \frac{1}{2}.$$

x y

الحل ⇐ $y = \dfrac{1}{x^2+C}$

a) $y(2) = \frac{1}{3}$

x y

$y = \dfrac{1}{x^2-1}$

$$y = \frac{1}{x^2+c} \Longrightarrow \frac{1}{3} = \frac{1}{4+c}$$

$$3 = 4 + c$$

$\boxed{C = -1}$

b) $y = \frac{1}{x^2+c}$ $y(-2) = \frac{1}{2}$

x   y

$\frac{1}{2} = \frac{1}{(-2)^2+c} \Rightarrow \frac{1}{2} = \frac{1}{4+c}$

$2 = 4+c$ $\boxed{c = -2}$

■ **Example 1.5** If $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

$$(a)\ y(0) = 1,\ y'(0) = 2\ \ (b)\ \ y(-1) = 5,\ \ y'(-1) = -5.$$

y	y'
e^x	e^x
e^{3x}	$3e^{3x}$
e^{-x}	$-e^{-x}$
e^{-2x}	$-2e^{-2x}$

a) $y = c_1e^x + c_2e^{-x}$

$y' = c_1e^x - c_2e^{-x}$

IV $\Rightarrow$ $y(0) = 1$

$y = c_1e^x + c_2e^{-x}$

$1 = c_1\cancel{e^0} + c_2\cancel{e^{-0}}$ $\boxed{c_1 + c_2 = 1}$

IV $\Rightarrow$ $y'(0) = 2$

$y' = c_1e^x - c_2e^{-x}$

$2 = c_1\cancel{e^0} - c_2\cancel{e^0}$ $\boxed{c_1 - c_2 = 2}$

$$c_1 + \cancel{c_2} = 1$$
$$c_1 - \cancel{c_2} = 2$$

$$2c_1 = 3 \qquad c_1 = \frac{3}{2} = 1.5$$

$$c_1 + c_2 = 1$$
$$1.5 + c_2 = 1 \qquad c_2 = 1 - 1.5 = \frac{-1}{2} = -0.5$$

$$\boxed{y = \frac{3}{2}e^{x} - \frac{1}{2}e^{-x}}$$

b) $y = c_1 e^{x} + c_2 e^{-x}$

$$y' = c_1 e^{x} - c_2 e^{-x}$$

$y(-1) = 5$

$$5 = \cancel{c_1} e^{-1} + c_2 e^{1}$$

$y'(-1) = -5$

$$-5 = \cancel{c_1} e^{-1} - c_2 e^{1}$$

$$\cancel{5} = c_1 e^{-1} + \cancel{c_2 e^{1}}$$
$$\cancel{-5} = c_1 e^{-1} - \cancel{c_2 e^{1}}$$

$$0 = 2c_1 e^{-1}$$
$$\boxed{0 = c_1}$$

$$5 = \cancel{c_1 e^{-1}} + c_2 e$$

$$\boxed{c_2 = 5e^{-1}}$$

$$y = 5e^{-1}e^{-x}$$

$$\boxed{y = 5e^{-(1+x)}}$$

Exercise 1.1 Verify that the indicated function is a solution of the given DE:
(i) $y'' - 4y' + 4y = 0; \quad y = c_1 e^{2x} + c_2 x e^{2x}$

$$y = c_1 e^{2x} + c_2 x e^{2x} = (c_1 + c_2 x)e^{2x}$$

$$y' = 2c_1 e^{2x} + 2c_2 x e^{2x} + c_2 e^{2x} = (2c_1 + 2c_2 x + c_2)e^{2x}$$

$$y'' = 4c_1 e^{2x} + 4c_2 x e^{2x} + 4c_2 e^{2x} = (4c_1 + 4c_2 x + 4c_2)e^{2x}$$

الان نعوض في المعادلة:

$$y'' - 4y' + 4y \stackrel{?}{=} 0$$

$$(4c_1 + 4c_2 x + 4c_2)e^{2x} - 4(2c_1 + 2c_2 x + c_2)e^{2x} + 4(c_1 + c_2 x)e^{2x} \stackrel{?}{=} 0$$

$$(\cancel{4c_1} + \cancel{4c_2x} + \cancel{4c_2} - \cancel{8c_1} - \cancel{8c_2x} - \cancel{4c_2} + \cancel{4c_1} + \cancel{4c_2x})e^{2x}$$

$$(0)e^{2x} = 0$$

$$0 = 0$$

LHS = RHS ✓

(ii) $y'' - 6y' + 13y = 0; \quad y = e^{3x}\cos 2x$

$$y = e^{3x}\cos 2x$$

$$y' = -2e^{3x}\sin 2x + 3e^{3x}\cos 2x$$

$$y'' = -4e^{3x}\cos 2x - 6e^{3x}\sin 2x - 6e^{3x}\sin 2x + 9e^{3x}\cos 2x$$

$$y'' = 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$$

$$y'' - 6y' + 13y \stackrel{?}{=} 0$$

$$5e^{3x}\cos 2x - 12e^{3x}\sin 2x - 6(-2e^{3x}\sin 2x + 3e^{3x}\cos 2x) + 13(e^{3x}\cos 2x)$$

$$5e^{3x}\cos 2x \cancel{- 12e^{3x}\sin 2x} \cancel{+ 12e^{3x}\sin 2x} - 18e^{3x}\cos 2x + 13e^{3x}\cos 2x$$

$$\cancel{18e^{3x}\cos 2x} - \cancel{18e^{3x}\cos 2x}$$

$$0 = 0$$

$$LHS = RHS$$

(iii) $y'' + y = \tan x; \quad y = -(\cos x)\ln(\sec x + \tan x)$

$$y = -\cos x \ln(\sec x + \tan x)$$

$$y' = -\cos x \left[\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\right] + \sin x \ln(\sec x + \tan x)$$

$$y' = -\cos x \left[\frac{\sec x \cancel{(\tan x + \sec x)}}{\cancel{\tan x + \sec x}}\right] + \sin x \ln(\sec x + \tan x)$$

$$\boxed{y' = -1 + \sin x \ln(\sec x + \tan x)}$$

$$y'' = \sin x \left[\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\right] + \cos x \ln(\sec x + \tan x)$$

$$\boxed{y'' = \tan x + \cos x \ln(\sec x + \tan x)}$$

الان نعوض في المعادلة

$$y'' + y \stackrel{?}{=} \tan x$$

$$\tan x + \cancel{\cos x \ln(\sec x + \tan x)} \cancel{- \cos x \ln(\sec x + \tan x)}$$

$$\tan x = \tan x$$

$$LHS = RHS$$

y	y'
$\ln x$	$\frac{1}{x}$
$\ln f$	$\frac{f'}{f'}$
$\ln(3x^2)$	$\frac{6x}{3x^2}$

y	y'
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$

$\sec x = \frac{1}{\cos x}$ $\quad$ $\csc x = \frac{1}{\sin x}$